University of Malta
Junior College

Subject: Advanced Pure Mathematics
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Time: 09.00 - 12.00

End of Year Test

Worked Solutions
Question 1

Given that ABCD is a parallelogram with coordinates: A(1,2), B(7,-1) and C(-1,-2).

a) To find the equation of lines AD and CD.

Let AD have equation \( y = m_1 x + c_1 \) and let CD have equation \( y = m_2 x + c_2 \).

Since AD is parallel to CB then they have same gradient. Similarly CD has the same gradient as AB. Thus,

\[
m_1 = \text{Gradient of CB} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-1 + 2}{7 + 1} = \frac{1}{8}
\]

\[
m_2 = \text{Gradient of AB} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-1 - 2}{7 - 1} = -\frac{1}{2}
\]

Line AD has gradient \( m_1 = \frac{1}{8} \) and passes through point A(1,2). We can use this information to find the y-intercept \( c_1 \) as follows:

\[
y = m_1 x + c_1
\]

\[
y = \frac{1}{8} x + c_1
\]

At point A, we have that \( x = 1 \) and \( y = 2 \) hence,

\[
y = \frac{1}{8} x + c_1
\]

\[
2 = \frac{1}{8} + c_1
\]

\[
\frac{15}{8} = c_1
\]

Thus AD has equation \( y = \frac{1}{8} x + \frac{15}{8} \), which can also be written as \( 8y = x + 15 \).
Line $CD$ has gradient $m_2 = -\frac{1}{2}$ and passes through point $C(-1,-2)$. We can use this information to find the y-intercept $c_2$ as follows:

$$y = m_2 x + c_2$$

$$y = -\frac{1}{2} x + c_2$$

At point $C$, we have that $x = -1$ and $y = -2$ hence,

$$y = -\frac{1}{2} x + c_2$$

$$-2 = \frac{1}{2} + c_2$$

$$-\frac{5}{2} = c_2$$

Thus $CD$ has equation $y = -\frac{1}{2} x - \frac{5}{2}$, which can also be written as $2y = -x - 5$.

b) To find the coordinates of point $D$. Point $D$ is the point of intersection of lines $AD$ and $CD$.

Line $AD$ has equation $8y = x + 15$ ... eqn. 1

Line $CD$ has equation $2y = -x - 5$ ... eqn. 2

We solve eqn. 1 and eqn. 2 simultaneously:

$$8y = x + 15$$

$$2y = -x - 5$$

We multiply eqn. 2 by 4

$$8y = x + 15$$

$$8y = -4x - 20$$

We subtract the two equations:

$$0 = 5x + 35$$

$$-35 = 5x$$

$$-7 = x$$

To find $y$, when $x = -7$:

$$2y = -x - 5$$

$$2y = 7 - 5$$

$$y = 1$$

Hence point $D$ has coordinates $(-7,1)$. 
c) To prove that \( \angle BAC \) is right-angled.

We consider the gradient of lines \( AB \) and \( AC \).

Gradient of line \( AB \) is \( m_2 = -\frac{1}{2} \).

Gradient of line \( AC \) is \( m_3 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-2 - 2}{-1 - 1} = 2 \).

These two lines are perpendicular if and only if \( m_2 \cdot m_3 = -1 \).

Since \(-\frac{1}{2} \times 2 = -1\), then \( AB \) is perpendicular to \( AC \) and hence \( \angle BAC \) is right-angled.

d) Area of parallelogram = Length of Base \( \times \) Perpendicular height.

Perpendicular height = Length of \( AC = \sqrt{(-1 - 1)^2 + (-2 - 2)^2} = 2\sqrt{5} \).

Length of Base = Length of \( AB = \sqrt{(7 - 1)^2 + (-1 - 2)^2} = 3\sqrt{5} \).

Area of parallelogram = \( 2\sqrt{5} \times 3\sqrt{5} = 6 \times 5 = 30 \) units\(^2\).

e) To find the shortest distance from \( A(1,2) \) to line \( BC \).

First we find equation of line \( BC \).

Gradient of line \( BC \) is \( \frac{1}{8} \). Thus line \( BC \) has equation \( y = \frac{1}{8}x + c \)

Line passes from point \((7,-1) \implies y = \frac{1}{8}x + c \)

\[
-1 = \frac{7}{8} + c \\
-1 - \frac{7}{8} = c \\
-\frac{15}{8} = c
\]

Hence \( BC \) has equation \( y = \frac{1}{8}x - \frac{15}{8} \), which can be written as \(-x + 8y + 15 = 0\).

Distance between \( A \) to line \( BC \) is \( \frac{|ah + bk + c|}{\sqrt{a^2 + b^2}} \). So,

\[
\left| \frac{-1(1) + 8(2) + 15}{\sqrt{(1+64)}} \right| = \frac{30}{\sqrt{65}} = 3.721
\]
Question 2

a) To resolve \( \frac{11x + 5}{(x - 2)(x + 1)^2} \) into partial fractions.

Let,

\[
\frac{11x + 5}{(x - 2)(x + 1)^2} \equiv \frac{A}{(x - 2)} + \frac{B}{(x + 1)} + \frac{C}{(x + 1)^2}.
\]

By L.C.M. we have that:

\[
11x + 5 \equiv A(x + 1)^2 + B(x - 2)(x + 1) + C(x - 2)
\]

Case 1 when \( x = 2 \):

\[
11(2) + 5 = A(2 + 1)^2 + B(2 - 2)(2 + 1) + C(2 - 2)
\]

\[
22 + 5 = 9A + 0 + 0
\]

\[
27 = 9A
\]

\[
3 = A
\]

Case 2 when \( x = -1 \):

\[
11(-1) + 5 = A(-1 + 1)^2 + B(-1 - 2)(-1 + 1) + C(-1 - 2)
\]

\[
-11 + 5 = 0 + 0 - 3C
\]

\[
-6 = -3C
\]

\[
2 = C
\]

Case 3 when \( x = 0 \):

\[
11(0) + 5 = A(0 + 1)^2 + B(0 - 2)(0 + 1) + C(0 - 2)
\]

\[
5 = A - 2B - 2C
\]

\[
5 = 3 - 2B - 4
\]

\[
5 = -1 - 2B
\]

\[
6 = -2B
\]

\[
-3 = B
\]

Answer:

\[
\frac{11x + 5}{(x - 2)(x + 1)^2} \equiv \frac{3}{(x - 2)} - \frac{3}{(x + 1)} + \frac{2}{(x + 1)^2}
\]
b) To show that \( \frac{6\left(2^{n+1}\right) - 4\left(2^{n-1}\right)}{2^{n+1} - 2^n} = 10 \)

Consider the left hand side:

\[
\frac{6\left(2^{n+1}\right) - 4\left(2^{n-1}\right)}{2^{n+1} - 2^n} = \frac{6\left(2^n \cdot 2\right) - 4\left(2^n \cdot 2^{-1}\right)}{2^n \cdot 2 - 2^n} = \frac{2^n (12 - 2)}{2^n} = 12 - 2 = 10
\]

**Question 3**

a) Given that \( f(x) \equiv x^3 + 5x^2 - 17x - 21 \)

(i) To find the remainder when \( f(x) \) is divided by \( (x - 3) \).

\[
f(x) \equiv x^3 + 5x^2 - 17x - 21
goalspace
f(3) \equiv (3)^3 + 5(3)^2 - 17(3) - 21
\equiv 27 + 45 - 51 - 21
\equiv 0
\]

Hence the remainder is 0.

To find the other factors, we use long division in algebra:

\[
\begin{array}{c|ccc}
& x^2 & + 8x & + 7 \\
\hline
(x - 3) & x^3 & + 5x^2 & - 17x - 21 \\
& x^3 & - 3x^2 & \\
\hline
& 8x^2 & - 17x & - 21 \\
& 8x^2 & - 24x & \\
\hline
& 7x & - 21 & \\
& 7x & - 21 & \\
\hline
& / & / & \\
\end{array}
\]

Hence \( f(x) \equiv (x - 3)(x^2 + 8x + 7) \)

\[\equiv (x - 3)(x + 7)(x + 1)\]
b) To solve the following simultaneous equations:

\[3^{x-y} = 9^y \quad \text{... eqn. 1}\]
\[2^x = 6(2^y) \quad \text{... eqn. 2}\]

Consider eqn. 1:

\[3^{x-y} = 9^y\]
\[\log_3 3^{x-y} = \log_3 9^y\]
\[(x-y)\log_3 3 = \log_3 3^{3y}\]
\[(x-y)\log_3 3 = 2y\log_3 3\]
\[x-y = 2y\]
\[x = 3y\]

Consider eqn. 2

\[2^x = 6(2^y)\]
\[2^{3y} = 6(2^y)\]
\[\log_2 2^{3y} = \log_2 6(2^y)\]
\[3y\log_2 2 = \log_2 6 + y\log_2 2\]
\[3y\log_2 2 - y\log_2 2 = \log_2 6\]
\[2y\log_2 2 = \log_2 6\]
\[y = \frac{1}{2}\log_2 6\]

So, \(y = \frac{1}{2}\log_2 6\). Recall that \(x = 3y\) so, \(x = \frac{3}{2}\log_2 6\).

**Answer:** \(x = \frac{3}{2}\log_2 6\)
\(y = \frac{1}{2}\log_2 6\)
**Question 4**

a) Given that \( \alpha \) and \( \beta \) are the roots of the quadratic equation \( 2x^2 + 3x + 4 = 0 \). So,

Sum of roots \( \Rightarrow \quad \alpha + \beta = \frac{-b}{a} = \frac{-3}{2} \)

Product of roots \( \Rightarrow \quad \alpha\beta = \frac{c}{a} = \frac{4}{2} = 2 \)

We are required to form a new quadratic equation whose roots are \( \frac{1}{\alpha^3} \) and \( \frac{1}{\beta^3} \).

Sum of roots \( \Rightarrow \quad \frac{1}{\alpha^3} + \frac{1}{\beta^3} = \frac{\alpha^3 + \beta^3}{(\alpha\beta)^3} \)

\[
= \frac{(\alpha + \beta)(\alpha^2 - \alpha\beta + \beta^2)}{(\alpha\beta)^3}
\]

\[
= \frac{(\alpha + \beta)((\alpha^2 + \beta^2) - \alpha\beta)}{(\alpha\beta)^3}
\]

\[
= \frac{(\alpha + \beta)((\alpha + \beta)^2 - 3\alpha\beta)}{(\alpha\beta)^3}
\]

\[
= \frac{(-\frac{3}{2})((-\frac{3}{2})^2 - 6)}{2^3}
\]

\[
= \frac{45}{64}
\]

Product of roots \( \Rightarrow \quad \frac{1}{\alpha^3} \cdot \frac{1}{\beta^3} = \frac{1}{(\alpha\beta)^3} \)

\[
= \frac{1}{2^3}
\]

\[
= \frac{1}{8}
\]

Thus new equation is \( x^2 - (\text{sum of roots})x + (\text{product of roots}) = 0 \)

\[
x^2 - \frac{45x}{64} + \frac{1}{8} = 0
\]

\[
64x^2 - 45x + 8 = 0
\]
b) To solve the following inequalities:

(i) Consider \(6x^2 - 7x > 3\)

\[
\begin{align*}
6x^2 - 7x &> 3 \\
6x^2 - 7x - 3 &> 0 \\
(3x + 1)(2x - 3) &> 0
\end{align*}
\]

Next we sketch the graph to identify the correct region as shown below:

From the graph it is evident that the inequality region is:

\[x < -\frac{1}{3} \text{ and } x > \frac{3}{2}\]

(ii) Consider \(\frac{x + 4}{2x - 3} < -1\)

\[
\begin{align*}
\frac{x + 4}{2x - 3} &< -1 \\
\frac{(x + 4)(2x - 3)}{(2x - 3)} &< -(2x - 3)^2 \\
(x + 4)(2x - 3) &< -(2x - 3)(2x - 3) \\
2x^2 - 3x + 8x - 12 &< -\left(4x^2 - 6x - 6x + 9\right) \\
2x^2 + 5x - 12 &< -4x^2 + 12x - 9 \\
6x^2 - 7x - 3 &< 0 \\
(3x + 1)(2x - 3) &< 0
\end{align*}
\]

From the graph it is evident that the inequality region is:

\[-\frac{1}{3} < x < \frac{3}{2}\]
Question 5

a) To eliminate $\theta$ from the following equations: $x = \cos 2\theta$ and $y = 2\cos \theta$.

Consider the first equation: $x = \cos 2\theta$

$$= 2\cos^2 \theta - 1.$$ 

But from the second equation we have that $\cos \theta = \frac{y}{2}$. If we substitute this into the equation above, we get:

$$x = 2\cos^2 \theta - 1$$

$$= 2\left(\frac{y}{2}\right)^2 - 1$$

$$x = \frac{y^2}{2} - 1$$

$$2(x + 1) = y^2$$

Answer: $2(x + 1) = y^2$.

b) To prove that $\sin 3x + 2\sin 5x \sin^2 x + \sin 7x = 2\sin 5x \cos^2 x$.

We start by considering the left hand side:

$$\sin 3x + 2\sin 5x \sin^2 x + \sin 7x = \sin 7x + \sin 3x + 2\sin 5x \sin^2 x$$

$$= 2\sin\left(\frac{7x + 3x}{2}\right)\cos\left(\frac{7x - 3x}{2}\right) + 2\sin 5x \sin^2 x$$

$$= 2\sin 5x \cos 2x + 2\sin 5x \sin^2 x$$

$$= 2\sin 5x \left(\cos 2x + \sin^2 x\right)$$

$$= 2\sin 5x \left(\cos^2 x - \sin^2 x + \sin^2 x\right)$$

$$= 2\sin 5x \cos^2 x$$
Question 6
Given that a circle has equation \( x^2 + y^2 - 5x - 5y + 10 = 0 \) and has centre \( C \).

a) To find the coordinates of the centre \( C \) and radius:

For a circle with equation \( x^2 + y^2 + 2gx + 2fy + c = 0 \), then the centre \( C \) is \((-g, -f)\) and the radius is \( \sqrt{g^2 + f^2 - c} \).

By comparison we have that:

\[
2g = -5 \quad \text{and} \quad 2f = -5
\]

\[
g = -\frac{5}{2} \quad \text{and} \quad f = -\frac{5}{2}
\]

Thus point \( C \) has coordinates \( \left( \frac{5}{2}, \frac{5}{2} \right) \).

The radius is \( r = \sqrt{g^2 + f^2 - c} \)

\[
= \sqrt{\frac{25}{4} + \frac{25}{4} - 10}
\]

\[
= \frac{\sqrt{10}}{2}
\]

b) Given that the line has equation \( y = mx \) and intersects with the circle.

(i) To prove that \( (1 + m^2)x^2 - 5(1 + m)x + 10 = 0 \).

**Proof:**

The equation of the circle is \( x^2 + y^2 - 5x - 5y + 10 = 0 \).

The equation of the line is \( y = mx \).

If we substitute the equation of the line into the equation of the circle we get:

\[
x^2 + (mx)^2 - 5x - 5(mx) + 10 = 0
\]

\[
x^2 + (mx)^2 - 5x - 5(mx) + 10 = 0
\]

\[
(1 + m^2)x^2 - 5(1 + m)x + 10 = 0
\]

As required.
(ii) To find the values of \( m \) for which \((1 + m^2)x^2 - 5(1 + m)x + 10 = 0\) has equal roots.

The above equation has equal roots when \( b^2 - 4ac = 0 \).

\[
b^2 - 4ac = 0
\]
\[
(-5(1 + m))^2 - 40(1 + m^2) = 0
\]
\[
25(1 + 2m + m^2) - 40 - 40m^2 = 0
\]
\[
25 + 50m + 25m^2 - 40 - 40m^2 = 0
\]
\[
-15 + 50m - 15m^2 = 0
\]
\[
3m^2 - 10m + 3 = 0
\]
\[
(3m - 1)(m - 3) = 0
\]

Hence \( m = 3 \) or \( m = \frac{1}{3} \).

(iii) The line would be a tangent to the given circle.

**Question 7**

Given that the functions \( f \) and \( g \) are defined as follows:

\[
f : x \rightarrow x^2, \ x \in \mathbb{R}
\]
\[
g : x \rightarrow \frac{4}{x - 1}, \ x \in \mathbb{R}, \ x \neq 1
\]

a) (i) To find the inverse \( g^{-1}(x) \), and state its domain.

Let ,

\[
y = \frac{4}{x - 1}
\]
\[
x - 1 = \frac{4}{y}
\]
\[
x = \frac{4}{y} + 1
\]

Hence \( g^{-1}(x) = \frac{4}{x} + 1 \) for all \( x \in \mathbb{R}, x \neq 0 \).
(ii) To find the composition of functions $f \circ g(x)$.

$$f \circ g(x) = f(g(x)) = \left( \frac{4}{x-1} \right)^2 = \frac{16}{(x-1)^2}$$

b)

(i)

(ii)

(iii)
Question 8

a) To differentiate the following with respect to $x$:

(i) $y = x^2e^{3x}$

Here we use the **product rule**:

Let, 

$u = x^2$ and $v = e^{3x}$

$u' = 2x$ and $v' = 3e^{3x}$

By the product rule we have that,

$$\frac{dy}{dx} = uv' + vu'$$

$$= 3x^2e^{3x} + 2xe^{3x}$$

$$= xe^{3x}(3x + 2)$$

**Answer** $\frac{dy}{dx} = xe^{3x}(3x + 2)$

(i) $y = \sin^2(2x)$

Here we will use the **chain rule**:

$$y = \sin^4(2x)$$

$$\frac{dy}{dx} = 4 \sin^3(2x) \cos(2x) \cdot 2$$

$$\frac{dy}{dx} = 8 \sin^3(2x) \cos(2x)$$

**Answer** $\frac{dy}{dx} = 8 \sin^3(2x) \cos(2x)$

b) Given that the curve $y = 2x^3 + ax^2 + bx + 4$ has a minimum point at $(1, -3)$.

First we find the values of $a$ and $b$.

By differentiation we have that: 

$$y = 2x^3 + ax^2 + bx + 4$$

$$\frac{dy}{dx} = 6x^2 + 2ax + b.$$ 

$$\frac{d^2y}{dx^2} = 12x + 2a$$

Since $(1, -3)$ is a point on the curve, then:

$$y = 2x^3 + ax^2 + bx + 4$$

$$-3 = 2 + a + b + 4$$

$$-9 = a + b \quad \text{... eqn. 1}$$
Given also that \((1,-3)\) is a minimum point,

\[
\frac{dy}{dx} = 6x^2 + 2ax + b
\]
\[
0 = 6 + 2a + b
\]
\[
-6 = 2a + b \quad \text{... eqn. 2}
\]

Next we solve eqn. 1 and eqn. 2 simultaneously:

\[
-9 = a + b
\]
\[
-6 = 2a + b
\]

By subtracting eqn. 2 from eqn.1 we have that:

\[
-3 = -a
\]
\[
a = 3
\]

We use eqn. 1 to find \(b\) when \(a = 3\).

\[
-9 = a + b
\]
\[
-9 = 3 + b
\]
\[
-12 = b
\]

**Answer** \(a = 3\) and \(b = -12\)

b) Before we sketch we need to find the other turning point:

\[
\frac{dy}{dx} = 6x^2 + 2ax + b
\]
\[
0 = 6x^2 + 6x - 12
\]
\[
0 = x^2 + x - 2
\]
\[
0 = (x-1)(x+2)
\]

Hence we have turning points when \(x = 1\) and \(x = -2\).

When \(x = -2\), \(y = 24\). Hence the other turning point is \((-2,24)\).

Next we determine the nature of the turning points:

At \((1,-3)\) \(\Rightarrow \) \[
\frac{d^2y}{dx^2} = 12(1) + 6 = 18 > 0 \]. Hence \((1,-3)\) is a minimum.

At \((-2,24)\) \(\Rightarrow \) \[
\frac{d^2y}{dx^2} = 12(-2) + 6 = -18 < 0 \]. Hence \((-2,24)\) is a maximum.

Now we can sketch the graph shown on the following page:
Question 9

To evaluate the following integral by substitution.

\[ \int_{0}^{2} \frac{\sqrt{1 + e^{-\frac{x}{2}}}}{e^{\frac{x}{2}}} \, dx \]

Given that:

\[ u = e^{-\frac{x}{2}} \]
\[ \frac{du}{dx} = -\frac{1}{2} e^{-\frac{x}{2}} \]
\[ -2e^{\frac{x}{2}} \, du = dx \]

when \( x = 0 \) \( \Rightarrow \) \( u = e^{0} = 1 \)

when \( x = 2 \) \( \Rightarrow \) \( u = e^{-1} \)

So,

\[ \int_{0}^{2} \frac{\sqrt{1 + e^{-\frac{x}{2}}}}{e^{\frac{x}{2}}} \, dx = -2 \int_{1}^{e^{-1}} \frac{\sqrt{1 + u}}{e^{\frac{u}{2}}} e^{\frac{x}{2}} \, du \]

\[ = -2 \int_{1}^{e^{-1}} \sqrt{1 + u} \, du \]

\[ = -2 \left[ \frac{2(1 + u)^{\frac{3}{2}}}{3} \right]_{1}^{e^{-1}} \]

\[ = -2 \left( \left[ \frac{2(1 + e^{-1})^{\frac{3}{2}}}{3} \right] - \left[ \frac{2(1 + 1)^{\frac{3}{2}}}{3} \right] \right) \]

\[ = 1.638 \]
Question 10

Let \( a \) be the first term and let \( r \) be the common ratio of a geometric progression.

First term \( \Rightarrow \) \( a \)
Second term \( \Rightarrow \) \( ar \)
Fourth term \( \Rightarrow \) \( ar^3 \)
Fifth term \( \Rightarrow \) \( ar^4 \)

Given that:

\[
a + ar = -4
a(1 + r) = -4 \quad \text{...eqn. 1}
\]

and

\[
ar^3 + ar^4 = 108
a(1 + r)r^3 = 108 \quad \text{...eqn. 2}
\]

a) To find the value of \( a \) and \( r \) we solve eqn. 1 and eqn. 2 simultaneously.

If we substitute eqn.1 into eqn.2 we get:

\[
-4r^3 = 108
r^3 = -27
r = -3
\]

By using eqn.1 we find the value of \( a \):

\[
a(1 + r) = -4
a(1 - 3) = -4
-2a = -4
a = 2
\]

**Answer:** \( a = 2 \) and \( r = -3 \).

b) To find the sum of terms from the fifth to the tenth term.

\[
\text{Required Sum} = \sum_{n=1}^{10} 2(-3)^{n-1} - \sum_{n=1}^{4} 2(-3)^{n-1}
\]

\[
= S_{10} - S_4
= \frac{2 \left( 1 - (-3)^{10} \right)}{1 + 3} - \frac{2 \left( 1 - (-3)^4 \right)}{1 + 3}
= -29484
\]

c) The series diverges because \( |r| = 3 > 1 \).