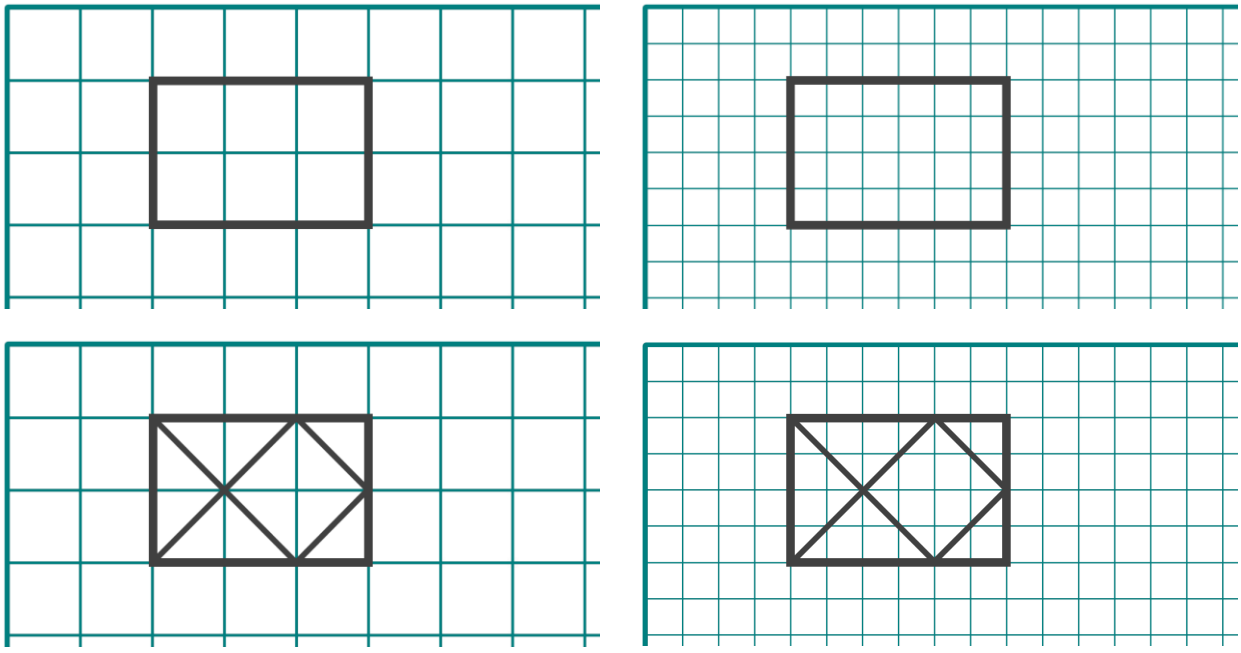
The Billiard Ball Solution

1. In which corner does the ball fall in each case?

a) The ball falls in the **bottom right corner** for the following table sizes:
2 by 1 4 by 2 4 by 3 6 by 5.
b) The ball falls in the **top right corner** for the following table sizes:
3 by 1 6 by 2 3 by 3 5 by 3 7 by 3.
c) The ball falls in the **top left corner** for the following table sizes:
3 by 2 5 by 2 6 by 4 5 by 4 7 by 4.

2. Group the cases where the ball falls in the same corner. What do they have in common? Can you see a pattern?

To see the pattern, we need to first remove those table sizes both of whose dimensions are even numbers: the 4 by 2, 6 by 2 and 6 by 4 tables respectively. These dimensions are equivalent to 2 by 1, 3 by 1 and 3 by 2 respectively, that is, to half the original measurements. The reason for this can be seen from the following diagram, where a 3 by 2 table is seen to be equivalent to a 6 by 4 table, simply by 'zooming in'.



After removing the table dimensions that have even numbers, we are left with the following dimensions:

a) The ball falls in the **bottom right corner** for the following table sizes:
   2 by 1, 4 by 3, 6 by 5.
b) The ball falls in the **top right corner** for the following table sizes:
   3 by 1, 3 by 3, 5 by 3, 7 by 3.
c) The ball falls in the **top left corner** for the following table sizes:
   3 by 2, 5 by 2, 5 by 4, 7 by 4.

What is the pattern?

In case a), the dimensions of the table are an **even number by an odd number**!

In case b), the dimensions of the table are an **odd number by an odd number**!

In case c), the dimensions of the table are an **odd number by an even number**!

3. *Without drawing anything, can you guess where the ball will fall in each of the following table sizes?*

   Armed with this easy pattern, we can now answer this question very quickly without drawing anything!

   The dimensions of the **10 by 9** table are an even number by an odd number, so we have case a): the ball falls in the **bottom right corner**.

   The dimensions of the **15 by 12** table are an odd number by an even number, so we have case c): the ball falls in the **top left corner**.

   The dimensions of the **33 by 17** table are an odd number by an odd number, so we have case b): the ball falls in the **top right corner**.

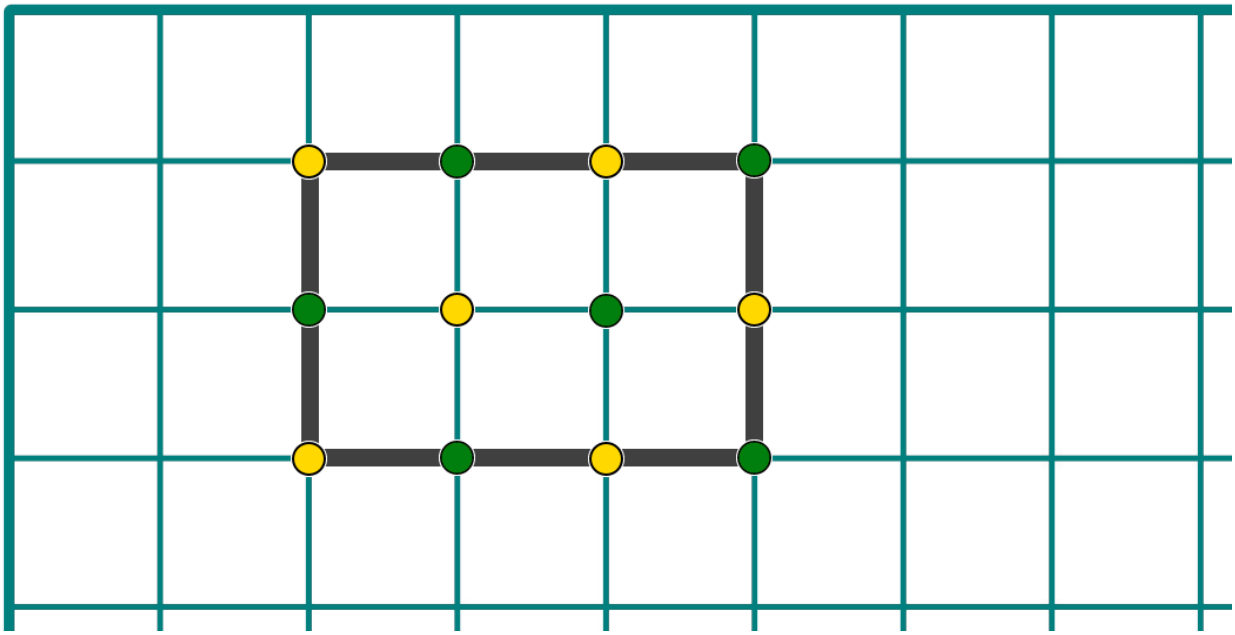
   The dimensions of the **142 by 105** table are an even number by an odd number, so we have case a): the ball falls in the **bottom right corner**.

   The dimensions of the **52 by 30** table are an even number by an even number. These are equivalent to half these dimensions, that is, 26 by 15, which are an even number by an odd number, so we have case a): the ball falls in the **bottom right corner**.

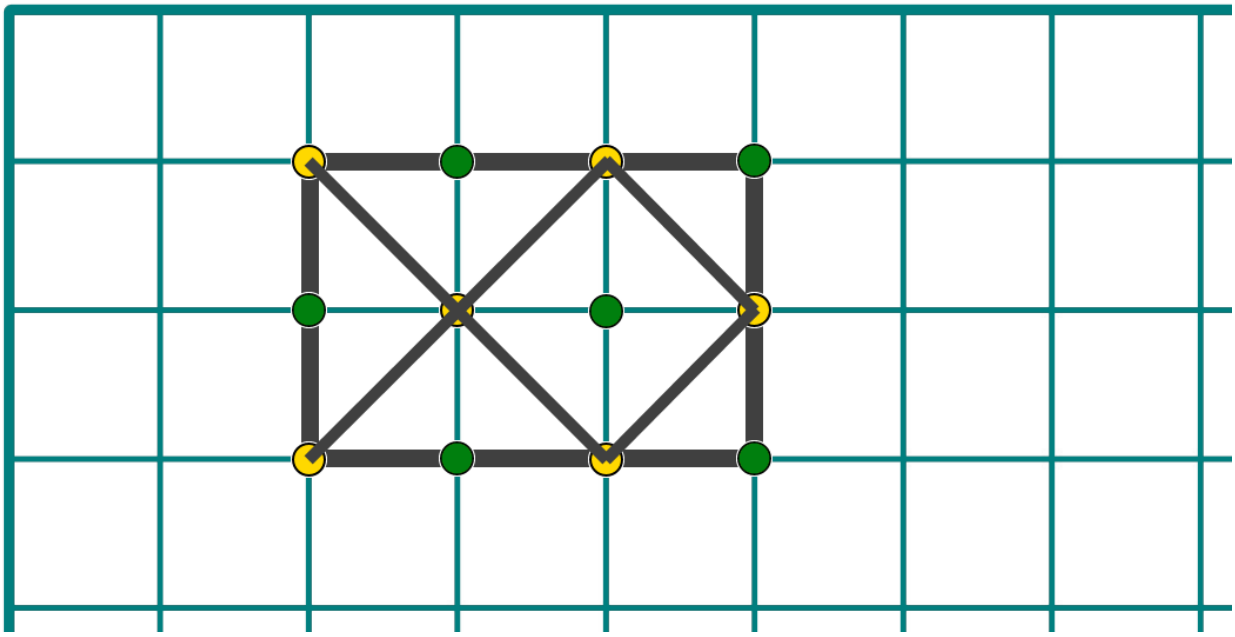
The dimensions of the **280 by 240** table are an even number by an even number. These are equivalent to half these dimensions, that is, 140 by 120. Again, the dimensions are both even, so we half them again, to 70 by 60. Since these dimensions are even again, we half them again to obtain 35 by 30, which are an odd number by an even number. So we have case c): the ball falls in the **top left corner**.

4. *Why does the pattern work?*

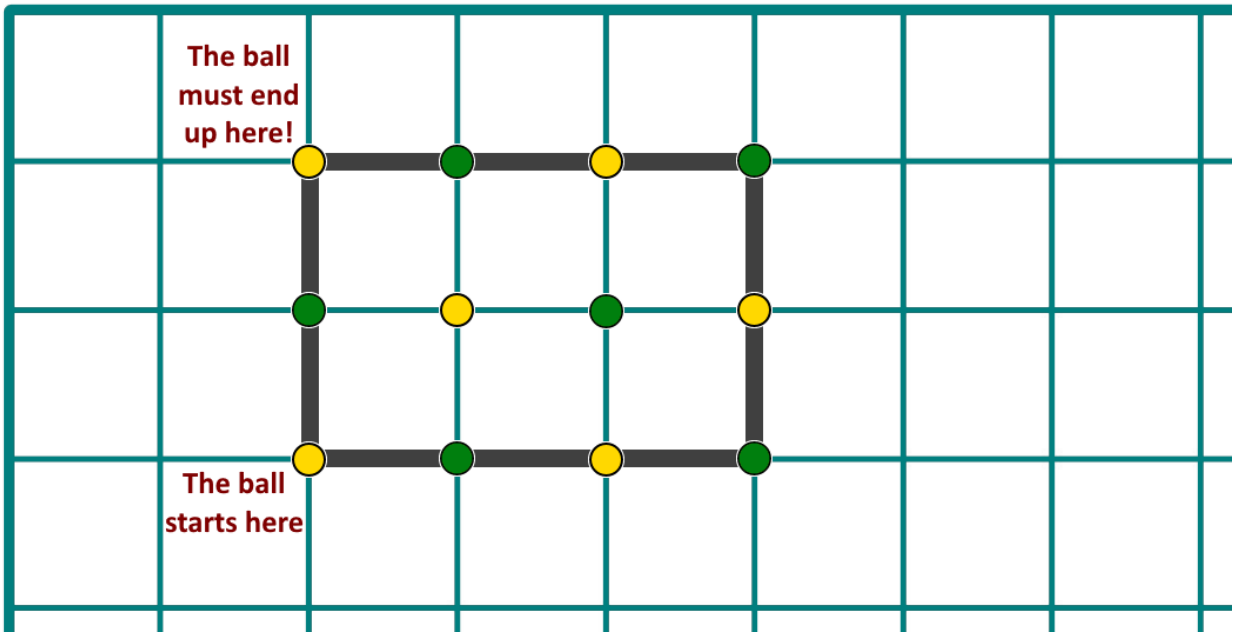
One way to see why this pattern works is by drawing the table on a squared paper, and colouring the intersections inside the table in two alternating dot colours. Like this:



It is clear that the lines we drew earlier will never go over any green dots, because they start from the bottom left corner, which is a yellow dot:



Because of this, we do not even need to draw the lines. We may just look at which corner has a yellow dot, besides the bottom left one. That corner is where the ball must end up in!



This explains why odd and even numbers are useful to understand the pattern. Moving from one coloured dot to another turns the number from odd to even and vice versa.

What else may I try to explore?

If you have reached this far and you are an older student, you may now try to find a pattern for the number of times the ball needs to bounce off the edge of the table in order to reach a corner. We shall not reveal the answer here, but there are many online resources that will surely help you. These include the following links:

A small program in Scratch that helps you solve the Billiard Ball Problem: https://scratch.mit.edu/projects/152682559/

The Billiard Ball Problem for Kids: https://mathforlove.com/wp-content/uploads/2017/04/Billiard-Ball-Problem.pdf

The Billiard Ball Problem for Slightly Older Kids: https://www.thehindu.com/children/the-billiard-ball-problem/article20314985.ece

Billiards for Big Kids! https://mathworld.wolfram.com/Billiards.html