Question 1

(a) \[ \frac{1}{4 + \sqrt{8}} + \frac{3}{4 - \sqrt{8}} = \frac{4 - \sqrt{8} + 3(4 + \sqrt{8})}{(4 + \sqrt{8})(4 - \sqrt{8})} \]
\[ = \frac{16 + 2\sqrt{8}}{16 - 8} \]
\[ = \frac{16 + 2\sqrt{8}}{8} = 2 + \frac{\sqrt{8}}{4}. \]
\[ \therefore a = 2, \ b = \frac{1}{4} \]

(b) \[ \frac{18}{x^3 - 9x} \equiv \frac{A}{x} + \frac{B}{x-3} + \frac{C}{x+3} \]

By taking the L.C.M., we get
\[ 18 \equiv A(x-3)(x+3) + Bx(x+3) + Cx(x-3) \]

Put \( x = 0 \) \quad \( 18 = -9A \quad \Rightarrow A = -2 \)

Put \( x = 3 \) \quad \( 18 = B(3)(6) \quad \Rightarrow B = \frac{1}{6} \)

Put \( x = -3 \) \quad \( 18 = C(-3)(-6) \quad \Rightarrow C = \frac{1}{2} \)

\[ \therefore \frac{18}{x^3 - 9x} \equiv \frac{-2}{x} + \frac{1}{x-3} + \frac{1}{x+3}. \]

Question 2

(a) \[ (ab)^2 + 2 = \left\{ \left(x^2 + x^{-1}\right)^2 \left(x^2 - x^{-2}\right) \right\}^2 + 2 \]
\[ = \left( x - x^{-1} \right)^2 + 2 \]
\[ = x^2 - 2 + x^2 + 2 = x^2 + x^{-2} \]
(b) \( f(x) \equiv x^3 + 2x^2 + 5x + p \)

\[ f(-3) \equiv (-3)^3 + 2(-3)^2 + 5(-3) + p = 0 \]

\[ -27 + 18 - 15 + p = 0 \Rightarrow p = 24 \]

Performing long division:

\[
\begin{array}{c|cccc}
\hline
& x^2 - x + 8 \\
\hline
x + 3 & x^3 + 2x^2 + 5x + 24 \\
\hline
& x^2 + 3x^2 \\
\hline
& -x^2 + 5x \\
\hline
& -x^2 - 3x \\
\hline
& 8x + 24 \\
\hline
\end{array}
\]

\[ 8x + 24 \]

\[ 8x + 24 \]

\[ \ldots \ldots \]

\[ \therefore \text{the quadratic factor is } x^2 - x + 8 \]

Question 3

(a) \( \log x^3 - \log xy \equiv \log \left( \frac{x^3}{xy} \right) \equiv \log \left( \frac{x^2}{y} \right) \).

(b) \( 3 \log x + 2 \log \left( \frac{1}{x} \right) \equiv \log x^3 + \log \left( \frac{1}{x^2} \right) \equiv \log \left( \frac{x^3}{x^2} \right) \equiv \log x. \)

(c) \( \frac{\log x^4 - \log x^2}{\log x^5 - \log x^3} \equiv \frac{\log \left( \frac{x^4}{x^2} \right)}{\log \left( \frac{x^5}{x^3} \right)} \equiv \frac{\log x^2}{\log x^2} \equiv 1. \)
Question 4

(a)  \((x-1)(x+k) = -4\)

\[ x^2 - x + kx - k + 4 = 0 \]
\[ x^2 + x(k - 1) + 4 - k = 0 \]

(i) For equal roots  \(b^2 - 4ac = 0\)

\[ (k-1)^2 - 4(4-k) = 0 \]
\[ k^2 - 2k + 1 - 16 + 4k = 0 \]
\[ k^2 + 2k - 15 = 0 \]
\[ (k+5)(k-3) = 0 \] \(\Rightarrow\)  \(k = -5, k = 3\)

(ii) For real, different roots \((b^2 - 4ac) > 0\)

Hence, we get \((k + 5)(k - 3) > 0\)

Solving this inequality, we get  \(k > 3, k < -5\)

(b) Let \(\alpha\) be one root, then \(\alpha^2\) is the other root.

Sum of roots = \(\alpha + \alpha^2 = \left(-\frac{b}{a}\right) = -\frac{a}{64}\)

Product of roots = \(\alpha (\alpha^2) = \left(\frac{c}{a}\right) = -\frac{27}{64}\).

The last equation reduces to \(\alpha^3 = -\frac{27}{64}\) \(\Rightarrow\)  \(\alpha = \sqrt[3]{-\frac{27}{64}} = -\frac{3}{4}\).

Substituting this value of \(\alpha\) in \(\alpha + \alpha^2 = -\frac{a}{64}\)

we get \(-\frac{3}{4} + \left(-\frac{3}{4}\right)^2 = -\frac{a}{64}\)

\(-\frac{3}{16} = -\frac{a}{64} \Rightarrow a = 12\)
Question 5

(a)  
(i) It is a function because it is one-to-one mapping  
(ii) It is not a function because it is many-to-many mapping  
(iii) It is a function because it is many-to-one mapping

(b)  
\[ f(x) \equiv x^2 + 1 \]
Since the coefficient of \( x^2 \) > 0, the quadratic function has a minimum point. Its minimum value of \( f(x) \) is 1 and this occurs when \( x = 0 \).

The range is: \( f(x) \geq 1 \)
Question 6

(i) The coordinates of A and D are \((-2, -1)\) and \((-1, 3)\) respectively.

\[
AD = \sqrt{(-2 + 1)^2 + (-1 - 3)^2} = \sqrt{(-1)^2 + (-4)^2} = \sqrt{17}.
\]

(ii) The gradient of \(AB\) is \(\frac{3 - (-1)}{6 - (-2)} = \frac{4}{8} = \frac{1}{2}\)

The gradient of \(DC\) is \(\frac{3 - 5}{-1 - 3} = \frac{-2}{-4} = \frac{1}{2}\)

Since the gradients are equal, then \(AB\) is parallel to \(DC\).

(iii) The gradient of the diagonal \(AC\) is \(\frac{5 - (-1)}{3 - (-2)} = \frac{6}{5}\)

The equation of the diagonal \(AC\) is \(y - y_1 = m(x - x_1)\)

\[
y - 5 = \frac{6}{5}(x - 3)
\]

\[
\Rightarrow 5y - 25 = 6(x - 3)
\]

\[
5y - 25 = 6x - 18
\]

\[
5y = 6x + 7
\]
The gradient of the diagonal BD = \( \frac{3 - (3)}{-1 - (6)} = 0 \)

The equation of the diagonal BD is \( y - y_1 = m(x - x_1) \)

\( y - 3 = 0(x - 6) \)

\[ \Rightarrow \quad y - 3 = 0 \]

\( y = 3 \)

Substitute \( y = 3 \) in \( 5y = 6x + 7 \)

\( 5(3) = 6x + 7 \)

\[ \Rightarrow \quad 6x = 8 \]

\( x = \frac{4}{3} \)

\[ : \quad \text{the point } E \text{ is } \left( \frac{4}{3}, 3 \right) \]

(iv) The midpoint of BD = \( \left( \frac{6 + (-1)}{2}, \frac{3 + 3}{2} \right) = \left( \frac{5}{2}, 3 \right) \)

The midpoint of AC = \( \left( \frac{-2 + (3)}{2}, \frac{-1 + 5}{2} \right) = \left( \frac{1}{2}, 2 \right) \)

Since none of the midpoints is the same as point E, then neither diagonals bisect each other.
Question 7

(a) \[ \sin^2 \theta = \frac{8}{9} \Rightarrow \sin \theta = \frac{\sqrt{8}}{3}, \] since \( \theta \) is obtuse.

Draw a right angled triangle:

\[ \sqrt{3^2 - 8} = 1 \]

The adjacent side was found by using Pythagoras’ theorem.

\[ \therefore \cos \theta = -\frac{1}{3} \]

The negative sign is included since \( \theta \) is obtuse.

(b)

\[
\begin{align*}
4 \sec^2 \theta &= 3 \tan \theta + 5 \\
4(\tan^2 \theta + 1) &= 3 \tan \theta + 5, & \text{since} & \sec^2 \theta = \tan^2 \theta + 1 \\
4 \tan^2 \theta + 4 &= 3 \tan \theta + 5 \\
4 \tan^2 \theta - 3 \tan \theta - 1 &= 0 \\
(4 \tan \theta + 1)(\tan \theta - 1) &= 0 \\
\tan \theta &= -\frac{1}{4}, 1
\end{align*}
\]
Use CAST rule to solve each equation:

\[\tan \theta = -\frac{1}{4}\]

we get \(\tan^{-1}\left(\frac{1}{4}\right) = 0.245\)

\[\therefore \theta = \pi - 0.245 = 2.897\]
\[\theta = 2\pi - 0.245 = 6.038\]

\[\tan \theta = 1\]

we get \(\tan^{-1}(1) = \frac{\pi}{4}\)

\[\therefore \theta = \frac{\pi}{4}\]
\[\theta = \pi + \frac{\pi}{4} = \frac{5\pi}{4}\]

Question 8

(a) \(y = (5 + 2 \ln x)^{-2}\)

\[
\frac{dy}{dx} = -2(5 + 2 \ln x)^{-3}\left(\frac{2}{x}\right) = -\frac{4}{x}(5 + 2 \ln x)^{-3}
\]
(b) \( y = \frac{e^{2x}}{2 - \sin 2x} \)

Use differentiation of a quotient: \( u = e^{2x} \quad v = 2 - \sin 2x \)

\[ u' = 2e^{2x} \quad v' = -2 \cos 2x \]

\[ \therefore \frac{dy}{dx} = \frac{(2 - \sin 2x)2e^{2x} - e^{2x}(-2 \cos 2x)}{(2 - \sin 2x)^2} \]

\[ = \frac{2e^{2x}(2 - \sin 2x + \cos 2x)}{(2 - \sin 2x)^2} \]

(c) \( y = x^2(x + 2)^{10} \)

Use differentiation of a product: \( u = x^2 \quad v = (x + 2)^{10} \)

\[ u' = 2x \quad v' = 10(x + 2)^9 \]

\[ \therefore \frac{dy}{dx} = x^210(x + 2)^9 + 2x(x + 2)^{10} \]

\[ = 2x(x + 2)^9(5x + x + 2) = 2x(x + 2)^9(6x + 2) = 4x(x + 2)^9(3x + 1). \]