Directions to candidates

Answer all questions. There are 10 questions in all.

The marks carried by each question are shown.

The total number of marks for all questions in the paper is 70.

Graphical calculators are NOT allowed.

Scientific calculators can be used, but all necessary working must be shown.

A booklet with mathematical formulae is provided.
1. \(ABCD\) is a parallelogram. The coordinates of \(A\), \(B\) and \(C\) are \((1, 2)\), \((7, -1)\) and \((-1, -2)\) respectively.

(i) Find the equation of the lines \(AD\) and \(CD\).
(ii) Hence, find the coordinates of the point \(D\).
(ii) Prove that angle \(BAC\) is rightangled.
(iv) Calculate the area of the parallelogram \(ABCD\).
(v) Find the shortest distance from \(A\) to \(BC\).

[2, 1, 1, 2, 2 marks]

2. (a) Resolve \(\frac{11x + 5}{(x - 2)(x + 1)^2}\) in partial fractions. [3 marks]

(b) Show that \(\frac{6(2^{n+1}) - 4(2^{n-1})}{2^{n+1} - 2^n} = 10\). [3 marks]

3. (a) A cubic polynomial is given by \(f(x) \equiv x^3 + 5x^2 - 17x - 21\).

(i) Find the remainder when \(f(x)\) is divided by \((x - 3)\).
(ii) Express \(f(x)\) as a product of three linear factors. [1, 2 marks]

(b) Solve the simultaneous equations

\[
\begin{align*}
3^x - y = 9^y \\
2^x = 6(2^y),
\end{align*}
\]

giving your answer in logarithmic form. [4 marks]

4. (a) \(\alpha\) and \(\beta\) are the roots of the quadratic equation \(2x^2 + 3x + 4 = 0\). Without solving the given equation, obtain the quadratic equation, with integer coefficients, whose roots are \(\frac{1}{\alpha^3}\) and \(\frac{1}{\beta^3}\). [4 marks]

(b) Solve the following inequalities:

(i) \(6x^2 - 7x > 3\),
(ii) \(\frac{x + 4}{2x - 3} < -1\). [2, 3 marks]
5. (a) Eliminate \( \theta \) from the equations

\[
    x = \cos 2\theta, \quad y = 2 \cos \theta.
\]

(b) Prove that \( \sin 3x + 2 \sin 5x \sin^2 x + \sin 7x \equiv 2 \sin 5x \cos^2 x \).

[2, 4 marks]

6. A circle with centre \( C \) has equation \( x^2 + y^2 - 5x - 5y + 10 = 0 \).

(a) Write down the coordinates of \( C \) and the radius of the circle.

(b) A line has equation \( y = mx \).

(i) Show that the \( x \)-coordinates of any points of intersection of the line and the circle satisfy the equation \( (1 + m^2)x^2 - 5(1 + m)x + 10 = 0 \).

(ii) Find the values of \( m \) for which the equation

\[
(1 + m^2)x^2 - 5(1 + m)x + 10 = 0
\]

has equal roots.

(iii) Describe the geometrical relationship between the line and the circle when \( m \) takes either of the values found in part (b) (ii).

[2, 1, 3, 1 marks]

7. The functions \( f \) and \( g \) are defined by

\[
    f : x \mapsto x^2, \quad x \in \mathbb{R}
\]

\[
    g : x \mapsto \frac{4}{x - 1}, \quad x \in \mathbb{R}, \ x \neq 1.
\]

(a) Find in its simplest form:

(i) the inverse function \( g^{-1} \), giving its domain,

(ii) the composite function \( f \circ g \).

[2, 1 marks]

(b) Sketch on separate diagrams the graphs of:

(i) \( y = f(x) \),

(ii) \( y = f(x) + 2 \),

(iii) \( y = -f(x) \).

[1 mark each]
8. (a) Differentiate each of the following with respect to $x$, simplifying your answers:

(i) $y = x^2 e^{3x}$, (ii) $y = \sin^4(2x)$.

$[2 \text{ marks each}]$

(b) Find $a$ and $b$ if the curve $y = 2x^3 + ax^2 + bx + 4$ has a minimum point at $(1, -3)$.

Hence, sketch the curve.

$[2, 3 \text{ marks}]$

9. Evaluate the integral $\int_0^2 \frac{\sqrt{1 + e^{-\frac{x^2}{2}}}}{e^{\frac{x^2}{2}}} \, dx$ by using the substitution $u = e^{-\frac{x^2}{2}}$.

Give your answer to 3 decimal places.

$[5 \text{ marks}]$

10. A geometric progression has the following properties:

the sum of the first and the second terms is $-4$, while the sum of the fourth and the fifth terms is $108$.

(i) Find the first term and the common ratio of this progression.

(ii) Find the sum of terms from the fifth term to the tenth term, both terms included.

(iii) Does this series converge or diverge? Explain why.

$[3, 3, 1 \text{ marks}]$