University of Malta

Junior College

Subject: Intermediate Pure Mathematics
Date: June 2010
Time: 09.00 - 12.00

End of Year Test

Worked Solutions
Question 1

(a) (i) \[ \frac{1}{1+\sqrt{2}} \times \frac{1-\sqrt{2}}{1-\sqrt{2}} = \frac{1-\sqrt{2}}{1-2} = \sqrt{2}-1 \]

(ii) \[ \frac{1}{\sqrt{2}+\sqrt{3}} \times \frac{\sqrt{2}-\sqrt{3}}{\sqrt{2}-\sqrt{3}} = \frac{\sqrt{2}-\sqrt{3}}{2-3} = \sqrt{3}-\sqrt{2} \]

(iii) \[ \frac{1}{\sqrt{3}+2} = \frac{1}{\sqrt{3}+2} \times \frac{\sqrt{3}-2}{\sqrt{3}-2} = \frac{\sqrt{3}-2}{3-4} = 2-\sqrt{3} \]

\[ \therefore \frac{1}{1+\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} + \frac{1}{\sqrt{3}+2} = \sqrt{2}-1+\sqrt{3}-\sqrt{2}+2-\sqrt{3} = 1 \]

(b) \[ \frac{2x-1}{(x+1)(x^2+2)} \equiv \frac{A}{(x+1)} + \frac{Bx+C}{(x^2+2)} \]

By taking the L.C.M, we get

\[ 2x-1 \equiv A(x^2+2)+(Bx+C)(x+1) \]

Put \( x = -1 \), \( 2(-1)-1 = A((-1)^2+2) \)

\[ \Rightarrow -3 = A(3) \Rightarrow A = -1 \]

Equate coefficients of \( x^2 \): \( 0 = A + B \)

\[ 0 = -1 + B \Rightarrow B = 1 \]

Equate constants:

\[ -1 = 2A + C \]

\[ -1 = -2 + C \Rightarrow C = 1 \]

\[ \therefore \frac{2x-1}{(x+1)(x^2+2)} \equiv \frac{-1}{(x+1)} + \frac{x+1}{(x^2+2)} \]
Question 2

(a) $E$ – midpoint of $AB = \left( \frac{6+0}{2}, \frac{7+5}{2} \right) = (3, 6)$

$F$ – midpoint of $BC = \left( \frac{2+0}{2}, \frac{1+5}{2} \right) = (1, 3)$

$G$ – midpoint of $CD = \left( \frac{12+2}{2}, \frac{2+1}{2} \right) = (7, 1.5)$

$H$ – midpoint of $AD = \left( \frac{12+6}{2}, \frac{2+7}{2} \right) = (9, 4.5)$
(b) \[ \text{gradient of } EH = \frac{4.5 - 6}{9 - 3} = \frac{-1.5}{6} = \frac{-1}{4} \]
\[ \text{gradient of } EF = \frac{6 - 3}{3 - 1} = \frac{3}{2} \]
\[ \text{gradient of } FG = \frac{3 - 1.5}{1 - 7} = \frac{1.5}{-6} = \frac{-1}{4} \]
\[ \text{gradient of } HG = \frac{4.5 - 1.5}{9 - 7} = \frac{3}{2} \]

(c) The gradient of \( EH = \) gradient of \( FG \)

The gradient of \( EF = \) gradient of \( HG \)

Thus the opposite sides of a quadrilateral are parallel.
Hence \( EFGH \) is a parallelogram.

Question 3

(a) \[ x^2 + k^2 = (k + 1)x \]
\[ x^2 - (k + 1)x + k^2 = 0 \]
\[ a = 1, \ b = -(k + 1), \ c = k^2 \]
For equal roots: \( b^2 = 4ac \)
\[ \therefore \ (k + 1)^2 = 4(k^2) \]
\[ k^2 + 2k + 1 = 4k^2 \]
\[ 0 = 3k^2 - 2k - 1 \]
\[ 0 = (3k + 1)(k - 1) \Rightarrow k = -\frac{1}{3}, \ 1 \]
(b) Sum of roots = \( \alpha + \beta = \frac{-b}{a} = 3 \)

Product of roots = \( \alpha \beta = \frac{c}{a} = 5 \)

Sum of new roots = \( \alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = 3^2 - 2(5) = -1 \)

Product of new roots = \( \alpha^2 \beta^2 = (\alpha\beta)^2 = 5^2 = 25 \)

Substituting in \( x^2 - (\text{Sum of roots})x + (\text{Product of roots}) = 0 \)

We get \( x^2 - (-1)x + (25) = 0 \)

i.e. \( x^2 + x + 25 = 0 \)

Question 4

(a) (i) \( f(x) \equiv x^3 - 2x^2 - 5x + 6 \)

\( f(1) \equiv 1^3 - 2(1)^2 - 5(1) + 6 = 0 \quad \Rightarrow \quad (x-1) \text{ is a factor} \)

(ii) \( f(-2) \equiv (-2)^3 - 2(-2)^2 - 5(-2) + 6 = 0 \quad \Rightarrow \quad (x+2) \text{ is a factor} \)

\( f(3) \equiv 3^3 - 2(3)^2 - 5(3) + 6 = 0 \quad \Rightarrow \quad (x-3) \text{ is a factor} \)

\( f(x) \equiv (x-1)(x+2)(x-3) \)

(iii) **Method 1**

\( g(x) \equiv x^3 - 3x^2 - x + 3 \)

\( g(1) \equiv 1^3 - 3(1)^2 - (1) + 3 = 0 \quad \Rightarrow \quad (x-1) \text{ is a factor} \)

\( g(-1) \equiv (-1)^3 - 3(-1)^2 - (-1) + 3 = 0 \quad \Rightarrow \quad (x+1) \text{ is a factor} \)

\( g(3) \equiv 3^3 - 3(3)^2 - (3) + 3 = 0 \quad \Rightarrow \quad (x-3) \text{ is a factor} \)

\( \therefore g(x) \equiv (x-1)(x+1)(x-3) \)
Method 2

\[ g(x) \equiv x^3 - 3x^2 - x + 3 \]

\[ g(x) \equiv x^3 - x - 3x^2 + 3 = x(x^2 - 1) - 3(x^2 - 1) = (x - 3)(x^2 - 1) \]

\[ \therefore g(x) \equiv (x - 3)(x - 1)(x + 1) \]

(iv) \[ \frac{f(x)}{g(x)} \equiv \frac{(x - 1)(x + 2)(x - 3)}{(x - 3)(x - 1)(x + 1)} \equiv \frac{x + 2}{x + 1} \]

(b) (i) \[ M = \log I \]

\[ 7.3 = \log I \Rightarrow I = 10^{7.3} = 19952623 \]

(ii) New Intensity = \[ 4 \times 19952623 = 79810492 \]

\[ \therefore M = \log 79810492 = 7.9 \]

Question 5

(a) \[ f(x) \equiv x^2 + 8x + 23 \]

\[ f(x) \equiv x^2 + 8x + 23 \equiv (x + a)^2 + b \equiv x^2 + 2ax + a^2 + b \]

Comparing coefficients of \( x \): \[ 8 = 2a \quad \Rightarrow a = 4 \]

Comparing constants: \[ 23 = a^2 + b \quad \Rightarrow 23 = 4^2 + b \quad \Rightarrow b = 7 \]
(b) \( f(x) \equiv (x + 4)^2 + 7 \)

Since the coefficient of \( x^2 \) > 0, the quadratic function has a minimum point. Minimum value of \( f(x) \) is 7 and this occurs when \((x + 4)^2 = 0\) i.e. \( x = -4 \)

Also when \( x = 0, \ f(x) = 23 \)

(c) By referring to the graph, the range of \( f \) is \( f(x) \geq 7 \)

Question 6

(a) \( 3x - 1 \leq x + 1 \)
\( 3x - x \leq 1 + 1 \)
\( 2x \leq 2 \ \Rightarrow \ x \leq 1 \)
(b) \( \frac{4-x}{x} > -2 \)

Multiplying both sides of the inequality by the denominator squared

\[
\left( \frac{4-x}{x} \right)^2 > -2x^2
\]

\( (4-x)x > -2x^2 \implies 4x-x^2 > -2x^2 \)

\[
x^2 + 4x > 0
\]

\[
x(x+4) > 0
\]

Solving this inequality, we get \( x > 0, x < -4 \)

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Question 7

(a) \( 6 \cos^2 \theta - \cos \theta = 1 \)

\[
6 \cos^2 \theta - \cos \theta -1 = 0
\]

\( (3 \cos \theta +1)(2 \cos \theta -1)=0 \implies \cos \theta = -\frac{1}{3} \text{ or } \frac{1}{2} \)

Use CAST rule to solve each equation:

\[
\implies \cos \theta = -\frac{1}{3}
\]

\[
\cos^{-1}\left(\frac{1}{3}\right) = 70.5^\circ
\]

\[
\therefore \theta = 180^\circ - 70.5^\circ \text{ or } 180^\circ + 70.5^\circ
\]

\[
\theta = 109.5^\circ \text{ or } 250.5^\circ
\]
(b) \[ \sec^4 \theta - \sec^2 \theta = \sec^2 \theta (\sec^2 \theta - 1) \]
\[ = \sec^2 \theta \tan^2 \theta \text{ by using the identity } \tan^2 \theta + 1 \equiv \sec^2 \theta \]
\[ = (\tan^2 \theta + 1) \tan^2 \theta = \tan^4 \theta + \tan^2 \theta \]

Question 8

(a) \[ y = x^3 + 4x^2 + 3x - 1 \]
\[ \frac{dy}{dx} = 3x^2 + 8x + 3 \]
At \( x = 1 \), the gradient of the tangent = \[ \frac{dy}{dx} = 3(1)^2 + 8(1) + 3 = 14 \]
Also at \( x = 1 \), \[ y = (1)^3 + 4(1)^2 + 3(1) - 1 = 7 \]
\[ \therefore \text{ equation of tangent: } y - 7 = 14(x - 1) \]
\[ \Rightarrow y = 14x - 7 \]
(b) 

(i) \( y = x^3(3 + 2x)^{12} \)

Use differentiation of a product:

\[
\begin{align*}
    u &= x^3 & v &= (3 + 2x)^{12} \\
    u' &= 3x^2 & v' &= 24(3 + 2x)^{11}
\end{align*}
\]

\[
\begin{align*}
    \frac{dy}{dx} &= 3x^2(3 + 2x)^{12} + x^3 \cdot 24(3 + 2x)^{11} \\
    &= 3x^2(3 + 2x)^{11}(3 + 2x + 8x) = 3x^2(3 + 2x)^{11}(3 + 10x)
\end{align*}
\]

(ii) \( y = \frac{e^x}{\sin 2x} \)

Use differentiation of a quotient:

\[
\begin{align*}
    u &= e^x & v &= \sin 2x \\
    u' &= e^x & v' &= 2\cos 2x
\end{align*}
\]

\[
\begin{align*}
    \frac{dy}{dx} &= \frac{e^x \sin 2x - e^x 2\cos 2x}{\sin^2 2x} \\
    &= \frac{e^x (\sin 2x - 2\cos 2x)}{\sin^2 2x}
\end{align*}
\]