Directions to candidates

Answer all questions. There are 10 questions in all.

The marks carried by each question are shown.

The total number of marks for all questions in the paper is 70.

Graphical calculators are NOT allowed.

Scientific calculators can be used, but all necessary working must be shown.

A booklet with mathematical formulae is provided.
1. (a) Find the values of the constants $a$ and $b$ if $(x + 1)$ and $(x - 2)$ are factors of the polynomial $f(x) \equiv x^3 + ax^2 + 2x + b$.

Find the third factor.

(b) Without using calculator, show that

$$\frac{\log_a \sqrt{64} + \log_a \sqrt{27} - \log_a \sqrt{125}}{\log_a 12 - \log_a 5} = \frac{3}{2},$$

where $a$ is a positive constant.

[3, 4 marks]

2. Let $f(x) \equiv 2x^2 - 8x + 9$.

(a) If $\alpha$ and $\beta$ are the roots of the equation $f(x) = 0$, form the quadratic equation with integer coefficients whose roots are $\alpha^2$ and $\beta^2$, without solving the given equation.

(b) By completing the square, show that $f(x) > 0$ for all real values of $x$.

[4, 3 marks]

3. (a) Resolve $\frac{4x^2 - 5x - 13}{x^2 - 2x - 3}$ in partial fractions.

(b)

The area of triangle $ABC$ is 10 cm$^2$. The length of the base $BC$ is $(\sqrt{5} + \sqrt{3})$ cm. Find the perpendicular height from $A$ to $BC$, expressing your answer in the form $a\sqrt{5} + b\sqrt{3}$, where $a$ and $b$ are integers.

[3, 2 marks]
4. (a) Show that \( \frac{\cos 2\theta}{\cos \theta + \sin \theta} \equiv \cos \theta - \sin \theta \), for \( \tan \theta \neq -1 \).

(b) Express \( \cos \theta - \sin \theta \) in the form \( R \cos(\theta + \alpha) \), where \( R > 0 \) and \( \alpha \) is acute.

(c) Hence by using the results of (a) and (b), solve correct to two decimal places, for \( 0^\circ \leq \theta^\circ \leq 360^\circ \):

\[
\frac{\cos 2\theta}{\cos \theta + \sin \theta} = \frac{1}{3}.
\]

[2, 3, 3 marks]

5. The midpoints of sides \( AB \), \( BC \) and \( AC \) of triangle \( ABC \) are \( D(2,1) \), \( E(4,3) \) and \( F(6,-1) \) respectively.

(i) Find the gradient of the line \( DE \).

(ii) Hence find the equation of the line \( AC \).

(iii) Find the area of triangle \( DEF \).

[1, 2, 2 marks]

6. Show that the locus of those points which are at a distance of \( \sqrt{20} \) from the point \( (1,-2) \) is a circle.

Hence find the equation of the tangent to the circle at the point \( (3,2) \).

[2, 3, marks]

7. The function \( f \) is defined by \( f : x \mapsto x^2 - 2x + 5 \), where \( x \geq -1 \).

(a) Sketch \( y = f(x) \) for the given domain.

(b) Explain why \( f^{-1}(x) \) does not exist.

How can the domain of \( f(x) \) be restricted so that \( f^{-1}(x) \) exists?

The function \( g \) is defined by \( g : x \mapsto 2x + 1 \), where \( x \geq -1 \).

(c) Find the composite function \( f \circ g \).

[3, 2, 2 marks]
8. Three numbers are three consecutive terms of an arithmetic progression. If their sum is 30, whilst their product is 750, find these three numbers. 

[5 marks]

9. (a) If \( x = t + \frac{1}{t}, \) \( y = 3t^2 - 2, \) show that \( \frac{dy}{dx} = \frac{6t^3}{t^2 - 1}. \)

Hence find \( \frac{d^2y}{dx^2} \) at \( t = 2. \)

(b) If the surface area of a sphere is decreasing at the rate of 4 cm\(^2\)/sec, find the rate of decrease of the radius when the radius of the sphere is 4 cm.

[3, 3, 3 marks]

10. (a) Find: (i) \( \int (x + 1)e^{x^2+2x} \, dx \) (ii) \( \int x^3 \ln 2x \, dx. \)

(b) Sketch on the same diagram the curves \( y = x^2 \) and \( y^2 = x. \)

Find the coordinates of the points where the two curves intersect.

Hence find the area enclosed between the two curves.

[2, 3, 2, 2, 3 marks]