University of Malta

Junior College

Subject: Intermediate Applied Mathematics
Date: June 2011
Time: 13.00 - 16.00

End of Year Test

Worked Solutions
Question 1

1. (a) By the Triangle Rule of Addition of Vectors: \( \mathbf{OB} = \mathbf{OA} + \mathbf{AB} = (a + 2x) \);
   \( \mathbf{OD} = \mathbf{OA} + \mathbf{AD} = (a + 2y) \);
   \( \mathbf{OC} = \mathbf{OA} + \mathbf{AB} + \mathbf{BC} = \mathbf{OA} + \mathbf{AB} + \mathbf{AD} = (a + 2x + 2y) \).

   (b) Taking \( d = \mathbf{OD}, \ g = \mathbf{OG} \) and so on,
   \( \mathbf{DG} = \mathbf{DO} + \mathbf{OG} = \mathbf{g} - \mathbf{d} = (a + x + y) - (a + 2y) = x - y \);
   \( \mathbf{GB} = \mathbf{GO} + \mathbf{OB} = \mathbf{b} - \mathbf{g} = (a + 2x) - (a + x + y) = x - y \).
   Hence \( \mathbf{DG} = \mathbf{GB} \) implying that \( DGB \) is one line and that \( G \) is the midpoint of diagonal \( BD \).
   Similarly, \( \mathbf{AG} = \mathbf{g} - \mathbf{a} = (a + x + y) - (a) = x + y \);
   \( \mathbf{GC} = \mathbf{c} - \mathbf{g} = (a + 2x + 2y) - (a + x + y) = x + y \).
   Hence \( \mathbf{AG} = \mathbf{GC} \) implying that \( G \) is the midpoint of diagonal \( AC \).
   Since \( G \) is on both diagonals of parallelogram \( ABCD \) it must be their point of intersection.

   (c) Taking \( \mathbf{OZ} = z \),
   \[ \text{L.H.S.} = \{\mathbf{a} - \mathbf{z}\} + \{(a + 2x) - \mathbf{z}\} + \{(a + 2x + 2y) - \mathbf{z}\} + \{(a + 2y) - \mathbf{z}\} \]
   \[ = 4a + 4x + 4y - 4z = 4\{(a + x + y) - \mathbf{z}\} = 4(\mathbf{g} - \mathbf{z}) = 4 \mathbf{ZG}. \]

   (d) \( \mathbf{PQ} = \mathbf{OQ} - \mathbf{OP} = (4i + 2j) - (i) = (3i + 2j) \);
   \( \mathbf{SR} = \mathbf{OR} - \mathbf{OS} = (3i + 4j) - (2j) = (3i + 2j). \)
   Hence \( \mathbf{PQ} = \mathbf{SR} \) implying that \( PQ \) and \( SR \) are equal and parallel so that \( PQRS \)
   is a parallelogram.
   Now \( \mathbf{PQ} = \mathbf{SR} = |3i + 2j| = (3^2 + 2^2)^{1/2} = 13^{1/2}. \)
   Also, \( \mathbf{RQ} = \mathbf{SP} = |\mathbf{OP} - \mathbf{OS}| = |i - 2j| = (5)^{1/2}. \)

   (e) From the result in (c) taking the origin at \( Z \) and the points \( A, B, C, D \) to be \( P, Q, R, S \),
   respectively we have \( \mathbf{OP} + \mathbf{OQ} + \mathbf{OR} + \mathbf{OS} = 4\mathbf{OG} \) so that the position
   vector of the point of intersection of the diagonals is \( \mathbf{g} = \mathbf{OG} = \frac{1}{4} (\mathbf{OP} + \mathbf{OQ} + \mathbf{OR} + \mathbf{OS}) = \frac{1}{4} \{(i) + (4i + 2j) + (3i + 4j) + (2j)\} = 2i + 2j. \)
Question 2

Referring to the diagrams on the left, since \( C \) is equidistant from \( A \) and \( B \),

\[
\tan \theta = \frac{25}{120} = \frac{5}{12}.
\]

So that \( \sin \theta = \frac{x}{5} \).

Then by Pythagoras’ Theorem, \( x = 13 \) and \( \sin \theta = \frac{5}{13}, \cos \theta = \frac{12}{13} \).

Equilibrium at \( A \) implies \( \text{resolving vertically}: \)

\[ T \sin \theta = 10 \text{N} \quad \text{or} \quad T\left(\frac{5}{13}\right) = 10, \quad \text{giving} \quad T = 13 \text{N}. \]

Equilibrium at \( B \) implies \( \text{resolving horizontally}: \)

\[ T \cos \theta = F \quad \text{or} \quad F = (26 \text{N})(\frac{12}{13}) = 24 \text{N}. \]

Resolving vertically:

\[ T \sin \theta + N_B = 70 \text{N} \quad \text{or} \quad N_B = 70 - (26)(\frac{5}{13}) = 60 \text{N}. \]

Hooke’s Law \( \Rightarrow \) \( T = \lambda e / a \)

\[ \Rightarrow \quad 26 = \lambda (AC + BC - 78) / 78 = \lambda (130 - 78) / 78 \]

\[ \Rightarrow \quad \lambda = (26)(78) / (52) = 39 \text{N}. \]

Question 3

3. (a) Since the 3 forces are in equilibrium, by the Principle of Moments, their sum of the moments, about any point, must add up to zero.

Hence, on taking moments about \( O \), the point of intersection of the lines of action of forces \( P \) and \( Q \),

we have:

\[ P (0) + Q (0) + R \text{ (perpendicular distance of } O \text{ from the line of action of } R) = 0. \]
Now since there are 3, not 2 forces, \( R \) cannot be the zero force. Hence the perpendicular distance of \( O \) from the line of action of \( R \) is zero. This implies that the line of action of \( R \) also passes through \( O \).

(b) Referring to the following diagrams, by Pythagoras’ Theorem,
\[
GE = (8^2 + 15^2)^{1/2} = 17\text{cm}.
\]
Similarly, we have \( FE = 25\text{cm} \).

If we consider the normal reaction, \( N_G \), and the frictional force, \( F_G \), of the peg at \( G \) on the lamina as a single resultant reaction \( R \), then the lamina may be perceived as being in equilibrium under the action of 3 non-parallel forces. These are: the tension in the string \( T \) acting on the lamina at \( E \), the weight of the lamina which also passes through \( E \) (since the lamina is uniform) and lastly the resultant reaction, \( R \), acting at \( G \). By the 3-Force Result, proved in (a), the line of action of \( R \) passes also through \( E \).

Hence \( R \) must be in the direction \( GE \) and makes angle \( \theta = \tan^{-1}(15/8) \) with the vertical.

This implies that in triangle \( FGE \), \( FG \) is vertical and hence parallel to the weight; \( EF \) is parallel to \( T \); and \( GE \) is parallel to \( R \).

Hence triangle \( FGE \) is a Triangle of Forces.

Since the forces are represented in magnitude and direction by a Triangle of Forces we have:
\[
\frac{FG}{W} = \frac{GE}{R} = \frac{EF}{T} \quad \text{or} \quad \frac{28\text{cm}}{56\text{N}} = \frac{17\text{cm}}{R} = \frac{25\text{cm}}{T} \quad \text{giving} \quad R = 34\text{N}, \quad T = 50\text{N}.
\]

The frictional force \( F \) is the horizontal component of \( R \) and has magnitude \( R \)
\[
\sin \theta = 34\text{N} \left( \frac{8}{17} \right) = 16\text{N}.
\]
Question 4

The forces keeping the system of rod plus particle in equilibrium are the normal reaction \( N_A \) and \( N_B \) at \( A \) and \( B \) respectively and the weights of the rod and small load. Their directions and lines of action are shown in the adjoining diagram.

Equilibrium of system

\[ \Rightarrow \text{resolving in direction of } N_A : \]
\[ N_A = (20 + 10) \text{ N} \cos \theta \]
\[ = 30 \text{ N} (3/5) = 18 \text{ N}. \]

\[ \text{resolving in direction of } N_B : \]
\[ N_B = (20 + 10) \text{ N} \sin \theta \]
\[ = 30 \text{ N} (4/5) = 24 \text{ N}. \]

\[ \text{taking moments about } B : \]
\[ 10N(x) + 20(NAB^2) = N_A(AB \cos \theta) \]
\[ 10x + 20 (100) = 18 (200 [3/5]) \]
\[ \text{or } x = 18 (12) 20 (10) = 16 \text{ cm}. \]

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Question 5

The geometry of the equilateral triangle \( ABC \) gives

\[ AO = (AB^2 - BO^2)^{1/2} = (16 - 4)^{1/2} = 2[3]^{1/2} \text{ m}. \]

Symmetry of triangle \( ABC \) implies that the perpendiculars from vertices \( B \) and \( C \) are also \( 2[3]^{1/2} \text{ m}. \)

From triangle \( BOX \), where \( X \) is the foot of the perpendicular from \( O \),

\[ d = BO \sin 60^0 = [3]^{1/2} \text{ m}. \]
For the resultant of any system of forces we have:

\[ (\sum \mathbf{F})_{\text{system}} = (\sum \mathbf{F})_{\text{resultant}} \]

and for a general point \( Z \), \( (\sum \text{moments about})_{\text{system}} = (\sum \text{moments about } Z)_{\text{resultant}} \)

(i) So when the system is in equilibrium,

\( (\text{moments about } A) \): \( P \cdot (AO) = 0 \text{ Nm} \)

\( \text{(resolving along BC)} \): \( P - (10 + Q) \cos 60^0 = 0 \text{ N} \) or \( 10 + Q = 0 \) giving \( Q = -10 \text{ N} \).

\( \text{(resolving along OA)} \): \( R + (\ Q - 10) \sin 60^0 = 0 \text{ N} \)

\( \text{giving } R = 10\sqrt{3} \text{ N} \).

(ii) When the resultant is a clockwise couple of moment \( 20\sqrt{3} \text{ Nm} \),

\( (\text{moments about } A) \): \( P \cdot (AO) = -20\sqrt{3} \) \text{ Nm} \) or \( P(2\sqrt{3}) = -20\sqrt{3} \)

\( \text{giving } P = -10 \text{ N} \).

\( \text{(resolving along BC)} \): \( P - (10 + Q) \cos 60^0 = 0 \text{ N} \) or \( -20 = 10 + Q \)

\( \text{giving } Q = -30 \text{ N} \).

\( \text{(resolving along OA)} \): \( R + (\ Q - 10) \sin 60^0 = 0 \text{ N} \)

\( \text{giving } R = 20\sqrt{3} \text{ N} \).

(iii) When the resultant is a force of \( 10\sqrt{3} \text{ N} \) acting at \( C \) in the direction \( OA \).

\( (\text{moments about } A) \): \( P \cdot (AO) = 10\sqrt{3} \cdot (OC) \) or \( P(2\sqrt{3}) = 10\sqrt{3} \) \text{ (2)}

\( \text{giving } P = 10 \text{ N} \).

\( \text{(resolving along BC)} \): \( P - (10 + Q) \cos 60^0 = 0 \text{ N} \) or \( 20 = 10 + Q \)

\( \text{giving } Q = 10 \text{ N} \).

\( \text{(resolving along OA)} \): \( R + (\ Q - 10) \sin 60^0 = 10\sqrt{3} \text{ N} \)

\( \text{giving } R = 10\sqrt{3} \text{ N} \).
Question 6

The \( v - t \) graph below represents the motion of the train from rest at station \( A \) to rest at station \( B \).

Since \( \text{gradient} \equiv \text{acceleration} \) and \( \text{area} \equiv \text{distance} \),
we have total area under graph \( = v \{(360+180)/2\} = 5,400 \text{ m} \)
giving \( v = 20 \text{ m s}^{-1} \)

Also, gradient in first 2 minutes is
\[
\frac{v}{120} = \frac{1}{6} \text{ms}^{-2}, \quad \text{giving the acceleration in}
\]
the first 2 minutes as required.

The gradient in the last minute is \( -\frac{v}{60} \) giving
the retardation in the last minute as \( \frac{1}{3} \text{ms}^{-2} \).

In the other voyage of the train from station \( A \) to \( B \), the train leaves \( A \) with maximum speed, \( v = 20 \text{ ms}^{-1} \) and only decelerates, as before, as it comes to a stop at \( B \). Hence the last part of this other journey must be similar to the first and takes exactly 60 s, while the first part must be covered with constant velocity \( v \).

Let the first part take \( T \) seconds. Then the \( v - t \) graph of the motion is as shown in
the adjoining diagram. Hence total area under graph \( = v \{(T + [T + 60])/2\} = 5,400 \text{ giving}
20 \{T + 30\} = 5,400 \quad \text{or} \quad T = (270 - 30) = 240 \text{s} \quad \text{so}
that the total time of the other journey is \( T + 60 = 300 \text{s} \quad \text{or} \quad 5 \text{ minutes} \).
Question 7

The Force-Acceleration diagrams of particles \( A \) and \( B \), while both particles are still in motion are shown below on the left. The common tension in the string and acceleration are symbolised by \( T \) (in Newton) and \( a \) (in \( \text{m s}^{-2} \)) whereas \( N \) is the normal reaction of the plane on \( A \).

Applying \( F = ma \), while both \( A \) and \( B \) are in motion, we get:

on \( A \), along the plane : \( T - 10 \sin 30^\circ = (1) a \) or \( T - 5 = a \)

on \( B \), vertically : \( 20 - T = (2) a \).

These equations give \( (T - 5) + (20 - T) = (1 + 2) a \) or \( a = 5 \) (m \( \text{s}^{-2} \)).

Since the acceleration is constant we may use the Equation for Constant Acceleration \( v^2 = u^2 + 2as \) for the particles’ motion until \( B \) hits the ground.

This gives: \( v^2 = 0^2 + 2 \times 5 \times 0.4 = 4 \) or \( v = 2 \). Thus the common speed of the particles when \( B \) hits the ground is \( 2 \text{ m s}^{-1} \).

When \( B \) hits the ground, the string becomes slack so that the only force acting on \( B \) besides the normal reaction is its weight of \( 10 \) N as shown in the diagram above on the right. Since the acceleration, \( f \), is along the plane in the direction of the resultant force, it points downwards, that is, in the opposite direction of the velocity of \( B \). This makes us consider \( B \)’s motion as that of retardation.

Applying \( F = ma \) during the entire motion of \( A \) while \( B \) is at rest:

on \( A \), along the plane : \( 20 \text{ N sin } 30^\circ = 2f \) giving \( f = 20/4 = 5 \text{ m s}^{-2} \).

Again, since the acceleration is constant, \( v^2 = u^2 + 2as \), applied from the moment \( B \) stops moving to the moment \( A \) comes to rest, gives: \( 0^2 = 2^2 + 2(-5)s \) leading to \( s = 4/10 \) or \( 0.4 \) m.

Since \( A \) had already moved an equal distance up the plane while \( B \) was in motion, that it has moved a total distance of \( 0.8 \) m, it reaches the pulley \( P \) just as it comes to rest.