University of Malta

Junior College

Subject: Advanced Pure Mathematics
Date: June 2011
Time: 09.00 - 12.00

End of Year Test

Worked Solutions
Question 1

(a) The gradient of the line $m = \frac{2 - 1}{6 - 2} = \frac{1}{4}$

Hence the equation of the line is $y - 1 = \frac{1}{4}(x - 2)$

$$4y - 4 = x - 2$$

4y - x = 2

The gradient of the line $2y = 6x + 7$ is 3

Let $\theta$ be the angle between the first line and the second line.

Then $\tan \theta = \frac{3 - \frac{1}{4}}{1 + 3\left(\frac{1}{4}\right)} = \frac{\frac{11}{4}}{\frac{7}{4}} = \frac{11}{7}$

$$\Rightarrow \theta = \arctan\left(\frac{11}{7}\right) = 57.53^0$$

(b) The line $\frac{x}{2a} + \frac{y}{b} = 1$ intersects the $x$-axis at P.

∴ putting $y = 0$, we get $x = 2a$. Point P $(2a, 0)$

The line intersects the $y$-axis at Q.

∴ putting $x = 0$, we get $y = b$. Point Q $(0, b)$

The distance $PQ = \sqrt{(2a - 0)^2 + (0 - b)^2} = \sqrt{4a^2 + b^2} = 3\sqrt{5}$

Squaring we get $4a^2 + b^2 = 45$ .... (i)

Gradient of PQ = $-\frac{b}{2a} = -\frac{1}{2} \quad \Rightarrow \quad b = a$ ....(ii)

Substituting (ii) in (i) we get $4a^2 + a^2 = 45$

$$\Rightarrow 5a^2 = 45 \quad \text{or} \quad a = 3$$, since $a > 0$

Using (ii) we get $b = 3$. 
Question 2

(a) \( f(x) \equiv 2x^3 + 5x^2 + ax + b \)
\[ f(-1) \equiv 2(-1)^3 + 5(-1)^2 + a(-1) + b = 0 \]
i.e. \(-2 + 5 - a + b = 0\) or \(b - a = -3\)…(i)
\[ f(2) \equiv 2(2)^3 + 5(2)^2 + a(2) + b = 36 \]
i.e. \(16 + 20 + 2a + b = 36\) or \(b + 2a = 0\) …(ii)
Solving simultaneously:
\[ b - a = -3 \]
\[ b + 2a = 0 \]
\[ -3a = -3 \quad \Rightarrow \quad a = 1 \]
Substituting in (i):
\[ b - 1 = -3 \quad \Rightarrow \quad b = -2 \]
\[ \therefore f(x) \equiv 2x^3 + 5x^2 + x - 2 \]
The second factor is found by trial and error.
\[ f(-2) \equiv 2(-2)^3 + 5(-2)^2 + (-2) - 2 = 0 \]
Hence \(x + 2\) is the second factor.
The third factor is found by inspection.
\[ \therefore 2x^3 + 5x^2 + x - 2 \equiv (x + 1)(x + 2)(cx + d) \]
Equating coefficients of \(x^3\):
\[ 2 = c \]
Equate constants
\[ -2 = 2d \quad \Rightarrow \quad d = -1 \]
\[ \therefore \text{the factors are } (x+1), (x+2) \text{ and } (2x-1). \]

(b) Sum of roots = \(\alpha + \beta = 3\)
Product of roots = \(\alpha\beta = 5\)
(i) \(\alpha^2 + \beta^2 \equiv (\alpha + \beta)^2 - 2\alpha\beta = 3^2 - 2(5) = -1.\)
(ii) \(\alpha^3 + \beta^3 \equiv (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta) = 3^3 - 3(5)(3) = -18.\)
The roots of the new equation are $\alpha^3 - \alpha^2$ and $\beta^3 - \beta^2$

Sum of roots $= \alpha^3 - \alpha^2 + \beta^3 - \beta^2 = \alpha^3 + \beta^3 - (\alpha^2 + \beta^2) = -18 - (-1) = -17$

Product of roots $= (\alpha^3 - \alpha^2)(\beta^3 - \beta^2) = (\alpha \beta)^3 - \alpha^2 \beta^3 - \alpha^3 \beta^2 + (\alpha \beta)^2$

$(\alpha \beta)^3 - \alpha^2 \beta^3 - \alpha^3 \beta^2 + (\alpha \beta)^2 = (\alpha \beta)^3 - (\alpha \beta)^2(\alpha + \beta) + (\alpha \beta)^2$

$= 5^3 - 5^2(3) + 5^2 = 75$

The equation is $x^2 - (-17)x + 75 = 0$ or $x^2 + 17x + 75 = 0$.

Question 3

(a) $x^2 + 2x + 9 > 12$

$x^2 + 2x - 3 > 0$

$(x + 3)(x - 1) > 0$

On solving this quadratic inequality we get $x > -1, x < -3$.

(b) $x^2 + 2x + 9 = kx$

$x^2 + x(2 - k) + 9 = 0$

For no real roots, we apply $b^2 - 4ac < 0$

$\therefore \quad (2 - k)^2 - 4(9) < 0$

$4 - 4k + k^2 - 36 < 0$

$k^2 - 4k - 32 < 0$

$(k - 8)(k + 4) < 0$

On solving this inequality we get $k < 8, k > -4$ or $-4 < k < 8$.

(c) $x^2 + 2x + 9 \equiv (x + A)^2 + B$

$x^2 + 2x + 9 \equiv x^2 + 2Ax + A^2 + B$

Equating coefficients of $x: 2 = 2A \Rightarrow A = 1$

Equating constants: $9 = A^2 + B$

$9 = 1 + B \Rightarrow B = 8$
\[ f(x) = (x+1)^2 + 8 \]

Least value of \( f(x) \) occurs when \((x + 1) = 0 \) i.e. at \( x = -1 \) and \( f(x) = 8 \)

Hence the greatest value of \( \frac{1}{f(x)} \) is \( \frac{1}{8} \).

(d) \[ y = f(x) = (x+1)^2 + 8 \]

It has a minimum at \((-1, 8)\) from part (c)

Also when \( x = 0, y = 9 \)

Sketching it, we get:

![Graph](image-url)
Question 4

(a) \( f(x) = \frac{12}{ax+b} \)

\[ f(-1) = \frac{12}{a(-1)+b} = -4 \quad \Rightarrow \quad 12 = 4a - 4b \quad \text{i.e.} \quad a - b = 3 \quad \ldots (i) \]

\[ f(2) = \frac{12}{a(2)+b} = 4 \quad \Rightarrow \quad 12 = 8a + 4b \quad \text{i.e.} \quad 2a + b = 3 \quad \ldots (ii) \]

Solving simultaneously,

\[ a - b = 3 \quad + \]
\[ 2a + b = 3 \]
\[ 3a = 6 \quad \Rightarrow \quad a = 2 \]

Substituting this value of \( a \) in (i) we get \( b = -1 \)

(b) Since \( f(x) = \frac{12}{2x-1} \), then the domain of \( f(x) \) is \( x \in \mathbb{R}, x \neq \frac{1}{2} \).

(c) Let \( y = f(x) = \frac{12}{2x-1} \)

Cross multiplication: \( 2xy - y = 12 \)

\[ 2xy = 12 + y \]

\[ x = \frac{12 + y}{2y} \]

\[ \therefore f^{-1}(x) = \frac{12 + x}{2x} \]

(d) \( f^{-1}(x) = \frac{12 + x}{2x} \)

If \( x = \frac{4}{5} \), then \( f^{-1}\left(\frac{4}{5}\right) = \frac{12 + \frac{4}{5}}{2\left(\frac{4}{5}\right)} = \frac{\frac{64}{5}}{\frac{8}{5}} = 8 \)
Question 5

(a) \(4 \sin x - 3 \cos x = R \sin(x - \alpha)\)

\[4 \sin x - 3 \cos x = R[\sin x \cos \alpha - \sin \alpha \cos x]\]

Equating coefficients of \(\sin x\): \(4 = R \cos \alpha\) .... (i)

Equating coefficients of \(\cos x\): \(3 = R \sin \alpha\) .......(ii)

\[(i)^2 + (ii)^2 \Rightarrow 4^2 + 3^2 = R^2(\cos^2 \alpha + \sin^2 \alpha)\]

\[25 = R^2, \text{ since } \left(\cos^2 \alpha + \sin^2 \alpha\right) = 1\]

\[\therefore R = 5, \text{ since } R > 0\]

\[
\frac{(ii)}{(i)} \Rightarrow \frac{3}{4} = \frac{R \sin \alpha}{R \cos \alpha}
\]

\[\therefore \tan \alpha = \frac{3}{4} \Rightarrow \alpha = \arctan\left(\frac{3}{4}\right) = 36.87^\circ\]

\[\therefore 4 \sin x - 3 \cos x = 5 \sin(x - 36.87^\circ)\]

(b) One of the formulae of the Factor Formulae is

\[2 \sin A \sin B \equiv \cos(A - B) - \cos(A + B)\]

\[\therefore 2 \sin x \sin(x - \alpha) \equiv \cos(\alpha) - \cos(2x - \alpha)\]

or \(\sin x \sin(x - \alpha) \equiv \frac{1}{2}\{\cos(\alpha) - \cos(2x - \alpha)\}\)

(c) \(5 \sin^2 x - 3 \sin x \cos x + \cos^2 x \equiv 5 \sin^2 x - 3 \sin x \cos x + 1 - \sin^2 x\)

\[\equiv 4 \sin^2 x - 3 \sin x \cos x + 1\]

\[\equiv \sin x(4 \sin x - 3 \cos x) + 1\]

\[\equiv \sin x\{5 \sin(x - 36.87^\circ)\} + 1, \text{ by using the result in (a)}\]

\[\equiv 5 \sin x \sin(x - 36.87^\circ) + 1\]

\[\equiv \frac{5}{2}\{\cos(36.87^\circ) - \cos(2x - 36.87^\circ)\} + 1, \text{ by using the result in (b)}\]

and using \(\alpha = 36.87^\circ\)

\[\equiv \frac{5}{2}\cos(36.87^\circ) - \frac{5}{2}\cos(2x - 36.87^\circ) + 1\]

\[\equiv 3 - \frac{5}{2}\cos(2x - 36.87^\circ)\]
\[ a = 3, \ b = -\frac{5}{2} \]

(d) \[ 5\sin^2 x - 3\sin x\cos x + \cos^2 x \equiv 3 - \frac{5}{2}\cos(2x - 36.87^\circ) \]

Since minimum value of the cosine function is \(-1\), then maximum of
\[ -\cos(2x - 36.87^\circ) = -1 \]
Substituting, we get \[ 3 - \frac{5}{2}(-1) = \frac{11}{2} \]

Question 6

(a) The general equation of a circle is \[ x^2 + y^2 + 2gx + 2fy + c = 0 \]
Substituting the point \((0, 2)\) : \[ 0 + 2^2 + 0 + 4f + c = 0 \]
This reduces to \[ 4f + c = -4 \ldots (i) \]
Substituting the point \((0, 9)\) : \[ 0 + 9^2 + 0 + 18f + c = 0 \]
This reduces to \[ 18f + c = -81 \ldots (ii) \]
\[ \therefore \ 4f + c = -4 \quad - \quad 18f + c = -81 \]

\[ -14f = 77 \Rightarrow f = \frac{77}{-14} = -\frac{11}{2} \]
Substituting it in (i), we get \[ 4\left(-\frac{11}{2}\right) + c = -4 \quad \Rightarrow \quad c = 18 \]
Substituting the point \((3, 0)\) in the general equation of the circle

\[ 3^2 + 0 + 6g + 0 + c = 0 \]
This reduces to \[ 6g + c = -9 \]

\[ 6g + 18 = -9 \quad \Rightarrow g = \frac{-27}{6} = -\frac{9}{2} \]
The equation of the circle is \[ x^2 + y^2 - 9x - 11y + 18 = 0 \].

Centre \( \left(\frac{9}{2}, \frac{11}{2}\right) \)

Radius = \[ \sqrt{g^2 + f^2 - c} = \sqrt{\left(-\frac{9}{2}\right)^2 + \left(-\frac{11}{2}\right)^2 - 18} = 5.7 \]
(b) The distance from the point \((6, 0)\) to the centre of the circle is
\[\sqrt{\left(6 - \frac{9}{2}\right)^2 + \left(0 - \frac{11}{2}\right)^2} = 5.7\]

\[\therefore\] Since this distance is equal to the radius of the circle, then the point \((6, 0)\) lies on the circumference of the circle.

Question 7

(a) (i) \(y = e^{2x} \cos x\)

Use differentiation of a product
\[u = e^{2x}\quad v = \cos x\]
\[u' = 2e^{2x}\quad v' = -\sin x\]

\[\therefore \quad \frac{dy}{dx} = \left(2e^{2x}\right)(\cos x) + \left(e^{2x}\right)(-\sin x)\]
\[= e^{2x}\{2\cos x - \sin x\} .\]

(ii) \(y = \frac{3x-1}{\sqrt{3x+2}}\)

Use differentiation of a quotient
\[u = 3x - 1\quad v = \sqrt{3x + 2}\]
\[u' = 3\quad v' = \frac{1}{2}(3x + 2)^{-\frac{1}{2}}3\]

\[\therefore \quad \frac{dy}{dx} = \frac{3\sqrt{3x+2} - (3x-1)\left(\frac{3}{2}\frac{(3x+2)^{\frac{1}{2}}}{(3x+2)}\right)}{(\sqrt{3x+2})^2}\]
\[= \frac{3\sqrt{3x+2} - \frac{3(3x-1)}{2\sqrt{3x+2}}}{2(3x+2)^{\frac{3}{2}}}\]
\[= \frac{3(3x+2)}{2(3x+2)^{\frac{3}{2}}} = \frac{18x + 12 - 9x + 3}{2(3x+2)^{\frac{3}{2}}} = \frac{9x + 15}{2(3x+2)^{\frac{3}{2}}} = \frac{3(3x + 5)}{2(3x+2)^{\frac{3}{2}}} .\]
(iii) \( y = \log_5 x \)

\[ y = \log_5 x = \frac{\log_e x}{\log_e 5}, \text{ using the change of base formula} \]

\[ \frac{dy}{dx} = \frac{1}{\log_e 5} \left( \frac{1}{x} \right) = \frac{1}{x \ln 5}. \]

(b)

With respect to the diagram, the surface area

\[ A = x^2 + 4xy, \text{ since we have an open box.} \]

\[ \therefore \quad 64 = x^2 + 4xy \]

\[ \Rightarrow \quad y = \frac{64 - x^2}{4x} \]

\[ = \frac{16}{x} - \frac{x}{4} \quad \ldots (i) \]

The volume \( V = x^2y \)

Substituting (i) in the above formula,

we get \( V = x^2 \left( \frac{16}{x} - \frac{x}{4} \right) \)

\[ V = 16x - \frac{x^3}{4} \]

\[ \frac{dV}{dx} = 16 - \frac{3x^2}{4} = 0, \text{ for maximum or minimum} \]

\[ \Rightarrow 16 = \frac{3x^2}{4} \quad \text{or} \quad x = \frac{8}{\sqrt{3}} \quad \text{(Note in this case} \ x = -\frac{8}{\sqrt{3}} \text{ is ignored)} \]

To check whether the found value of \( x \) is a maximum or minimum.

\[ \frac{d^2V}{dx^2} = -\frac{6x}{4} \]

Substituting \( x = \frac{8}{\sqrt{3}} \), we get \( \frac{d^2V}{dx^2} < 0 \)

\[ \therefore \text{ for maximum volume} \ x = \frac{8}{\sqrt{3}}. \]
Question 8

(a) \( y = x - 1 + \frac{1}{x+1} = x - 1 + (x+1)^{-1} \)

\[ \frac{dy}{dx} = 1 - (x+1)^{-2} = 0 \text{ for maximum or minimum ...(i)} \]

\[ \Rightarrow 1 = (x+1)^{-2} = \frac{1}{(x+1)^2} \]

\( (x+1)^2 = 1 \)

Square root: \( x + 1 = \pm 1 \)

\[ \Rightarrow \quad x = 0 \text{ or } -2 \]

To find the y coordinate, substitute each value of x in \( y = x - 1 + \frac{1}{x+1} \)

\[ \therefore \quad \text{when } x = 0, \quad y = 0 - 1 + 1 = 0 \]

\[ \quad \text{when } x = -2, \quad y = -2 - 1 - 1 = -4 \]

The turning points are \((0, 0)\) and \((-2, -4)\)

Nature of turning points: Differentiate again equation (i)

\[ \frac{d^2y}{dx^2} = 2(x+1)^{-3} \]

Substituting \( x = 0 \) in \( \frac{d^2y}{dx^2} \), we get that \( \frac{d^2y}{dx^2} > 0 \)

\[ \therefore (0, \ 0) \text{ is a minimum point} \]

Substituting \( x = -2 \) in \( \frac{d^2y}{dx^2} \), we get that \( \frac{d^2y}{dx^2} < 0 \)

\[ \therefore (-2, -4) \text{ is a maximum} \]
(b) Area = \( \int_a^b y \, dx \)

\[
= \int_0^4 \left( x - 1 + \frac{1}{x + 1} \right) \, dx
\]

\[
= \left[ \frac{x^2}{2} - x + \ln|x + 1| \right]_0^4
\]

\[
= \frac{4^2}{2} - 4 + \ln 5 - 0 + 0 - \ln 1
\]  

\[
= 4 + \ln 5
\]

Question 9

(a) \( \int_1^e \left( \frac{\ln x}{x} \right)^2 \, dx \)

Integration by substitution is used

Let \( u = \ln x \)

\[
\frac{du}{dx} = \frac{1}{x} \Rightarrow dx = x \, du
\]

Change of limits

If \( x = 1 \), then \( u = \ln 1 = 0 \)

If \( x = e \), then \( u = \ln e = 1 \)

\[
\therefore \int_0^1 \frac{u^2}{x} \, x \, du = \int_0^1 u^2 \, du = \left[ \frac{u^3}{3} \right]_0^1 = \frac{1}{3} - 0 = \frac{1}{3}
\]

(b) \( \frac{x^2 + x - 11}{(x - 2)^3(x - 3)} \equiv \frac{A}{x - 2} + \frac{B}{(x - 2)^2} + \frac{C}{x - 3} \)

Take the L.C.M.

\[
x^2 + x - 11 \equiv A(x - 2)(x - 3) + B(x - 3) + C(x - 2)^3
\]

Put \( x = 2 \): \( 2^2 + 2 - 11 = B(2 - 3) \quad \Rightarrow -5 = -B \quad \text{i.e.} \quad B = 5 \)
Put $x = 3$: \( 3^2 + 3 - 11 = C(3 - 2)^2 \Rightarrow 1 = C \)

Equate coefficients of $x^2$: \( 1 = A + C \)

\[ 1 = A + 1 \Rightarrow A = 0 \]

\[ \therefore \frac{x^2 + x - 11}{(x-2)^2(x-3)} = \frac{5}{(x-2)^2} + \frac{1}{x-3} \]

\[
\int_4^6 \frac{x^2 + x - 11}{(x-2)^2(x-3)} \, dx = \int_4^6 \left( \frac{5}{(x-2)^2} + \frac{1}{x-3} \right) \, dx
\]

\[
= \int_4^6 \left( 5(x-2)^{-2} + \frac{1}{x-3} \right) \, dx
\]

\[
= \left[ \frac{5(x-2)^{-1}}{-1} + \ln |x-3| \right]_4^6
\]

\[
= \left[ \frac{5(6-2)^{-1}}{-1} + \ln |6-3| - \frac{5(4-2)^{-1}}{-1} - \ln |4-3| \right]
\]

\[
= \left[ -\frac{5}{4} + \ln 3 + \frac{5}{2} - \ln 1 \right] = \frac{5}{4} + \ln 3
\]

Question 10

(a) Since \( S_6 = 42 \)

we have \( \frac{6}{2} \{ 2a + (6-1)d \} = 42 \)

\[ 3 \{ 2a + 5d \} = 42 \]

\[ 2a + 5d = 14 \ldots (i) \]

Also 2\text{nd} term = 7(5\text{th} term)

\[ \therefore \quad a + d = 7(a + 4d) \]

\[ a + d = 7a + 28d \]

\[ -6a = 27d \quad \text{or} \quad a = -\frac{27}{6}d = -\frac{9}{2}d \]
Substituting in (i), we get
\[
2 \left( -\frac{9}{2} \right) + 5d = 14
\]

\[
= -9d + 5d = 14
\]

\[
= -4d = 14 \quad \text{or} \quad d = -\frac{14}{4} = -\frac{7}{2}
\]

Substituting, we get
\[
a = -\frac{9}{2} \left( -\frac{7}{2} \right) = 63 \frac{3}{4}
\]

(b) The terms 21 + x, 27 + x and 29 + x are consecutive terms of a G.P.

\[
∴ \text{the common ratio } r = \frac{27 + x}{21 + x} = \frac{29 + x}{27 + x}
\]

Cross multiplication: 
\[
(27 + x)^2 = (29 + x)(21 + x)
\]

Expanding: 
\[
729 + 54x + x^2 = 609 + 50x + x^2
\]

\[
729 - 609 = 50x - 54x
\]

\[
120 = -4x
\]

\[
\Rightarrow \quad x = -30
\]