Directions to candidates

Attempt any 7 questions.

The marks carried by each question are shown.

The total number of marks for all questions in the paper is 100.

Graphical calculators are NOT allowed.

Scientific calculators can be used, but all necessary working must be shown.

A booklet with mathematical formulae is provided.

Where necessary take $g=10\text{ms}^{-2}$
1. Forces of $(8\mathbf{i} + 4\mathbf{j}) \text{ N}$, $(6\mathbf{i} + 5\mathbf{j}) \text{ N}$, $(-2\mathbf{i} - 2\mathbf{j}) \text{ N}$ and $-2\mathbf{j} \text{ N}$ act at the points with coordinates $(0, 0)$, $(0, 3)$, $(3, 4)$ and $(4, 0)$ respectively, where distances are measured in metres.

(i) Find the magnitude and direction of the resultant force $\mathbf{F}$ of this system of forces.

(ii) Show that the line of action of $\mathbf{F}$ cuts the $y$-axis at the point $(0, 2)$ and hence obtain its Cartesian equation.

(iii) This system of forces is equivalent to a force acting at the origin together with a couple $C$. Write down the magnitude of $C$ and indicate its sense on a diagram.

[3, 4, 3 marks]

2. A uniform solid circular cylinder is in equilibrium with one plane face on a rough inclined plane. The plane is inclined to the horizontal at an angle $\alpha$ degrees, which can be varied. The cylinder has a weight $W$, diameter $d$ and height $3d$.

(i) Draw a diagram showing all the forces acting on the cylinder, clearly indicating the position of the line of action of the normal reaction of the inclined plane on the cylinder.

(ii) If the plane is sufficiently rough to prevent sliding, find the maximum value of $\alpha$ for the cylinder to remain in equilibrium.

(iii) The coefficient of friction between the cylinder and the plane is $\frac{2}{9}$. If the value of $\alpha$ is gradually increased from zero, show that the cylinder will slide before it topples.

[2, 4, 4 marks]
3. The diagram shows the velocity-time graph for a particle $P$ which travels on a straight line $AB$, where $v$ m s$^{-1}$ is the velocity of $P$ at time $t$ s. The graph consists of five straight line segments. The particle starts from rest when $t = 0$ at a point $X$ on the line between $A$ and $B$ and moves towards $A$. The particle comes to rest at $A$ when $t = 2.5$ s.

(i) Given that the distance $XA$ is 4 m, find the greatest speed reached by $P$ during this stage of motion.

In the second stage, $P$ starts from $A$ when $t = 2.5$ s and moves towards $B$. The distance $AB$ is 48 m. The particle takes 12 s to travel from $A$ to $B$ and comes to rest at $B$. For the first 2 s of this motion $P$ accelerates at 3 m s$^{-2}$, reaching a velocity $V$ m s$^{-1}$. Find:

(ii) the value of $V$;
(iii) the value of $t$ at which $P$ starts to decelerate during this stage;
(iv) the deceleration of $P$ immediately before it reaches $B$.

[3, 2, 4, 1 marks]

4. A car moves on a horizontal circular path of radius 60 m around a bend that is banked at an angle $\tan^{-1}\frac{1}{24}$ to the horizontal. If the coefficient of friction between the tyres of the car and the road surface is $\frac{3}{5}$, find the greatest speed with which the car can be driven without any slipping occurring.

[10 marks]
5. A uniform rod $AB$ of weight $W$ has its end $A$ on a rough horizontal ground and rests at $45^\circ$ to the vertical against a smooth peg at $C$, where $AC = \frac{3}{4}AB$.

(i) If the rod is on the point of slipping in the vertical plane containing the rod, find $\mu$, the coefficient of friction between the rod and the ground.

(ii) A weight $W$ is attached at $B$. Show that the line of action of the resultant of this weight and that of the rod passes through $C$. Hence, or otherwise, deduce that in this case the coefficient of friction required to maintain equilibrium is $\mu = 1$.

[7, 3 marks]

6. (a) Anne has made a simple pendulum which beats every 2 seconds.
State the period and frequency of the pendulum and show that its length is $\frac{4g}{\pi^2}$.
(You may use the formula, relating the period of the simple pendulum with its length, without proof.)

(b) A particle moves with simple harmonic motion on a straight line between the points $P$ and $Q$. The amplitude of this motion is 0.3 m.
When the particle is 0.06 m from $P$, its speed is 0.9 m s$^{-1}$.

(i) Show that the motion of the particle is characterised by the 2nd order differential equation $\ddot{x} = -25x$.

(ii) Hence show that the period of motion is $\frac{2\pi}{5}$ s.

(iii) Find the magnitude of the maximum velocity and maximum acceleration of the particle.

[2; 4, 2, 2 marks]
7. A framework is composed of seven light smoothly jointed rods $AB$, $AE$, $BE$, $BD$, $ED$, $BC$ and $DC$, so that $ABDE$ is a square and $BDC$ is a right-angled triangle. The rod $AB$ has length $l$ and angle $\angle CBD = 45^\circ$.

The framework is in a vertical plane and is freely hinged at $A$ to a fixed support. A load of 10,000 N is attached to $C$. The rod $AE$ is kept vertical by a smooth support at $E$.

(i) Show that the reaction of the support at $E$ on the framework is 20,000 N.

(ii) Find the magnitude of the reaction force on the framework at $A$ and the angle it makes with the horizontal.

(iii) Find the magnitude of the force in each rod, stating whether each rod is in tension or in compression.

[1, 2, 7 marks]

8. A light elastic string of natural length 0.3 m has one end fixed to a point on the ceiling. To the other end of the string is attached a particle of mass $m$. When the particle is hanging freely in equilibrium, the length of the string is 0.4 m.

(i) Find the modulus of elasticity of the string in terms of $m$ and $g$.

A horizontal force is applied to the particle so that it is held in equilibrium with the string making an angle $\alpha$ with the downward vertical. The length of the string is now 0.45 m.

(ii) Show that $\alpha \approx 48.2^\circ$.

(iii) Find the horizontal force and the energy stored in the string.

[3, 4, 3 marks]
9. A small block $P$ is attached to another small block $Q$ by a light inextensible string. The block $P$ rests on a rough horizontal surface and the string passes over a smooth pulley so that $Q$ hangs freely, as shown in the diagram.

![Diagram of blocks $P$ and $Q$](image)

The block $P$ has a mass of 0.4 kg and the coefficient of friction between $P$ and the surface is 0.5. The block $Q$ has a mass if 0.3 kg. The system is released form rest and $Q$ moves vertically downwards.

(i) Draw a diagram to show the forces acting on $P$ and $Q$.

(ii) Show that the frictional force between $P$ and the surface has a magnitude of 2 N.

(iii) By forming an equation of motion for each block, show that the magnitude of the acceleration of each block is $1\frac{3}{7}$ m s$^{-2}$.

(iv) Find the speed of the blocks after 7 s of motion (assuming there is enough space for each block to continue with its motion uninterrupted).

(v) If after 7 s of motion the string suddenly breaks, comment on how would the speed of each block change.

[2, 1, 3, 2, 2 marks]

10. With its engine working at a constant rate of 10 kW, a car of mass 800 kg can descend a slope, inclined at $\sin^{-1} \frac{1}{40}$, at twice the steady speed that it can ascend the same slope, the resistance to motion remaining the same throughout.

(i) Find the magnitude of the resistance and the speed of ascent.

(ii) If after ascending the slope mentioned above, the car reaches a level road, assuming that during the ascent it reached the maximum speed found in part (i) and that the resistance to motion is the same as before, find the initial acceleration of the car.

[8, 2 marks]
11. A golf ball is struck from a point O with a velocity of \(24\, \text{m s}^{-1}\) at an angle of \(30^\circ\) to the horizontal. The ball first hits the ground at a point \(P\), which is at a height \(h\) metres above the level of \(O\).

The horizontal distance between \(O\) and \(P\) is 36 m.

(i) Show that the time the ball takes to travel from \(O\) to \(P\) is \(\sqrt{3}\) s.

(ii) Find the value of \(h\).

(iii) Find the speed with which the ball hits the ground at \(P\).

(iv) Find the angle between the direction of motion and the horizontal as the ball hits the ground at \(P\).

[2, 4, 3, 1 marks]

12. Three smooth spheres \(A\), \(B\), and \(C\) of equal radii and masses \(m\), \(m\) and \(2m\) respectively lie at rest on a smooth horizontal table. The centres of the spheres lie in a straight line with \(B\) between \(A\) and \(C\). The coefficient of restitution between any two spheres is \(e\). The sphere \(A\) is projected directly towards \(B\) with speed \(u\) and collides with \(B\).

(i) Show that the speed of \(B\) immediately after the impact between \(A\) and \(B\) is given by \(\frac{u}{2}(1 + e)\).

(ii) The sphere \(B\) subsequently collides with \(C\). The speed of \(C\) immediately after this collision is \(\frac{3}{8}u\). Find the value of \(e\).

[4, 6 marks]
13. A cyclist and his bicycle have a combined mass of 81 kg. The cyclist starts from rest in a straight line. He exerts a constant force of 135 N and motion is opposed by a resistance of magnitude $9v$ N, where $v \text{ m s}^{-1}$ is the speed of the cyclist at time $t$ s after the starting.

(i) Write the equation of motion and hence obtain the differential equation

$$\frac{9}{15 - v} \frac{dv}{dt} = 1.$$

(ii) Solve the differential equation to show that $v = 15(1 - e^{-\frac{t}{9}})$. Hence show that after enough time has elapsed the cyclist reaches a maximum speed $\approx 15 \text{ m s}^{-1}$.

(iii) Find the distance travelled by the cyclist in the first 9 s of the motion.

[3, 4, 3 marks]

14. The unit vectors $\mathbf{i}$ and $\mathbf{j}$ are directed due east and due north respectively. Two cyclists, Mario and Harvey, are cycling on straight horizontal roads with constant velocities of $(6\mathbf{i} + 2\mathbf{j}) \text{ km h}^{-1}$ and $(12\mathbf{i} - 8\mathbf{j}) \text{ km h}^{-1}$ respectively. Initially, Mario and Harvey have position vectors $(5\mathbf{i} - \mathbf{j}) \text{ km}$ and $(18\mathbf{i} + 5\mathbf{j}) \text{ km}$ respectively, relative to a fixed origin.

(i) Find, as vectors in terms of $\mathbf{i}$ and $\mathbf{j}$, the velocity and the initial position of Harvey relative to Mario.

(ii) Hence show that in any subsequent time $t$ hours after they start, the position of Harvey relative to Mario is given by

$$\mathbf{r}_H - \mathbf{r}_M = (13 + 6t)\mathbf{i} + (6 - 20t)\mathbf{j}.$$

(iii) Using the dot product, or otherwise, find the time of closest approach between Mario and Harvey.

[2, 4, 4 marks]