University of Malta

Junior College

Subject: Advanced Applied Mathematics
Date: June 2013
Time: 9.00 - 12.00

End of Year Test

Worked Solutions
At Equilibrium:

Resolving vertically

\[ R + 2W \sin 30^\circ = 4W \]

\[ R + 2W \left( \frac{1}{2} \right) = 4W \]

\[ R + W = 4W \quad \Rightarrow \quad R = 3W \]

Resolving horizontally

\[ \mu R = 2W \cos 30^\circ \]

\[ \mu (3W) = 2W \left( \frac{\sqrt{3}}{2} \right) \]

\[ \Rightarrow \quad \mu = \frac{\sqrt{3}}{3} = \frac{1}{\sqrt{3}} \]

Angle of friction = \( \lambda = \tan^{-1}(\mu) = \tan^{-1}\left( \frac{1}{\sqrt{3}} \right) = 30^\circ \)
With respect to the diagram above:

Resolving \( \downarrow \quad R = 10 \text{ N} \)

\[ \Leftrightarrow \quad T = \mu R = \left( \frac{1}{2} \right)(10) = 5 \text{ N} \]

Hence \( T = 5 \text{ N} \)

Applying Hooke’s law i.e. \( T = \frac{\lambda x}{a} \),

where \( a \) is the natural length = 20 cm = 0.2 m, \( \lambda \) is modulus of elasticity and \( x \) is the extension.
we get \[ 5 = \frac{50x}{0.2} \]

Hence \[ x = \frac{5(0.2)}{50} = 0.02 \text{ m} \]

The above applies for the closest and furthest positions. In the former it is a compression, while in the latter it is an extension.

\[ \therefore \text{ Furthest point} = 0.2 + 0.02 = 0.22 \text{ m from support.} \]

\[ \text{Closest point} = 0.2 - 0.02 = 0.18 \text{ m from support.} \]

Question 2

(i)

The pump is 60% efficient. Hence the actual mechanical work is

\[ 60\% \text{ of } 0.825 \text{ kW} = \left( \frac{60}{100} \right) \times 0.825 = 0.495 \text{ kW} = 495 \text{ W} \ldots \text{(a)} \]

The volume of water per minute = 0.3 m³.
Hence the volume per second \(= \frac{0.3}{60} = \frac{1}{200} \text{ m}^3 = 0.005 \text{ m}^3\)

Since Density = \(\frac{\text{Mass}}{\text{Volume}}\), then Mass = Density \(\times\) Volume

\[= 1000 \left(\frac{1}{200}\right) = 5 \text{ kg/s}\]

Power = \(\frac{\text{Energy}}{\text{s}} = \frac{\text{K.E.} + \text{P.E.}}{\text{s}}\)

\[\text{Energy/s} = \frac{1}{2}mv^2 + mgh\]

By (a) above, \(\text{Energy/s} = 495 \text{ J, } m = 5 \text{ kg/s, } h = 5 \text{ m}\)

Substituting, we get

\[495 = \frac{1}{2}(5)v^2 + 5(10)5\]

On simplifying

\[495 - 250 = 2.5v^2\]

\[\Rightarrow \quad 495 - 250 = 2.5v^2\]

Hence

\[v^2 = 98 \quad \text{i.e. } v = \sqrt{98} = 9.9 \text{ m/s}\]

(ii)

If \(V\) is the velocity of water through the nozzle and

\(A\) is the cross sectional area of the nozzle,

Then volume per second = \(AV\)

\[0.005 = A(9.9)\]

Hence \(A = \frac{0.005}{9.9} = 0.00051 \text{ m}^2 = 5.1 \text{ cm}^2\).
(i) Consider the block $B$.

Given $u = 0, \ t = 2, \ s = 1, \ a = ?$

Applying the equation of motion: $s = ut + \frac{1}{2}at^2$

$$1 = 0(2) + \frac{1}{2}a(2)^2$$

$$1 = \frac{1}{2}a(2)^2 \Rightarrow a = \frac{1}{2} \text{ m/s}^2$$

Applying: $v = u + at$

$$v = 0 + \frac{1}{2}(2) \Rightarrow v = 1 \text{ m/s}$$
(ii) Applying Newton’s second law $F = ma$ on block $B$,

\[ 25 - T = 2.5 \left( \frac{1}{2} \right) \]

\[ 25 - T = 1.25 \quad \Rightarrow \quad T = 25 - 1.25 = 23.75 \, \text{N} \]

At equilibrium,

Resolving $\downarrow$ on block $A$ \quad $R = 100 \, \text{N}$

Applying Newton’s second law $F = ma$ on block $A$,

\[ T - \mu R = 10a \]

Substituting \quad $23.75 - \mu(100) = 10 \left( \frac{1}{2} \right)$

\[ 23.75 - \mu(100) = 5 \quad \Rightarrow \quad \mu = 0.1875 \]

(iii) After $B$ hits the floor, there is no tension $T$ in the string.

Applying Newton’s second law $F = ma$ on block $A$,

\[ \mu R = 10a \]

Substituting \quad $0.1875(100) = 10a$

\[ \Rightarrow \quad a = 1.875 \, \text{m/s}^2 \quad \text{decelerating} \]

Applying the equation of motion: $v^2 = u^2 + 2as$

\[ 0 = (1)^2 + 2(-1.875)s \]

\[ 0 = 1 - 3.75s \]

Hence \quad $s = \frac{1}{3.75} = 0.267 \, \text{m}$
Question 4

(a) Applying the equations of motion

\[ \uparrow \quad s = 0 \]
\[ \quad u = V \]
\[ \quad a = -g \]

\[ \therefore \quad \text{substituting in} \quad s = ut + \frac{1}{2}at^2 \]
\[ 0 = Vt - \frac{1}{2}gt^2 \]
\[ 0 = t \left( V - \frac{1}{2}gt \right) \]

\[ \Rightarrow \quad t = 0, \quad \text{or} \quad t = \frac{2V}{g} \]
Considering in horizontal direction:

\[ s = R \]
\[ u = U \]
\[ g = 0 \]

On substituting in the same equation of motion, we get \( R = Ut \)

Hence \( R = U \left( \frac{2V}{g} \right) = \frac{2UV}{g} \)

For maximum height, time of flight = \( \frac{1}{2} \left( \frac{2V}{g} \right) = \frac{V}{g} \)

Applying the equations of motion for maximum height

\[ s = H \]
\[ u = V \]
\[ a = -g \]
\[ t = \frac{V}{g} \]

\[ s = ut + \frac{1}{2}at^2 \]

\[ H = V \left( \frac{V}{g} \right) - \frac{1}{2}g \left( \frac{V}{g} \right)^2 = \frac{V^2}{2g} \]
Let $U$ be the initial speed

In the horizontal direction $\rightarrow$ \[ s = 20 \]
\[ u = U \]
\[ a = 0 \]
\[ \therefore \text{using } s = ut + \frac{1}{2}at^2, \text{ we get } 20 = Ut \quad \ldots(i) \]

In the vertical direction $\downarrow$ \[ s = 2.5 \]
\[ u = 0 \]
\[ a = 10 \]
\[ \therefore \text{using } s = ut + \frac{1}{2}at^2, \text{ we get } 2.5 = 0 + \frac{1}{2}(10)t^2 \]
\[ 2.5 = 5t^2 \text{ i.e. } t^2 = \frac{1}{2} \text{ or } t = \frac{1}{\sqrt{2}} \]

Substituting in (i), we get $20 = U\left(\frac{1}{\sqrt{2}}\right) \Rightarrow U = 20\sqrt{2} \text{ m/s}$
In the horizontal direction \( \rightarrow s = 12 \)

\[
\begin{align*}
u &= U = 20\sqrt{2} \\
a &= 0
\end{align*}
\]

\[\therefore \text{using } s = ut + \frac{1}{2}at^2, \text{ we get } 12 = 20\sqrt{2} t \quad \Rightarrow \quad t = \frac{12}{20\sqrt{2}} = \frac{3}{5\sqrt{2}}\]

This is the time that the ball is over the net.

In the vertical direction \( \downarrow t = \frac{3}{5\sqrt{2}} \)

\[
\begin{align*}
u &= 0 \\
a &= 10
\end{align*}
\]

\[\therefore \text{using } s = ut + \frac{1}{2}at^2, \text{ we get } s = 0 + \frac{1}{2}(10)\left(\frac{3}{5\sqrt{2}}\right)^2 = 5\left(\frac{9}{50}\right) = 0.9 \text{ m}\]

This implies that the ball descended a distance of 0.9 m from the starting position, which is 25 m above ground.

\[\Rightarrow \text{ the ball is } 2.5 - 0.9 = 1.6 \text{ m above ground.}\]
Question 5

(i) Let $R$ and $R_1$ be the reactions at points $A$ and $C$, as shown.

Taking moments at $A$:

$$\frac{3l}{2} R_1 = l W \cos 60^\circ$$

Thus $R_1 = \frac{W}{3}$

Resolving vertically, we get

$$R + R_1 \cos 60^\circ = W \quad \Rightarrow \quad R + R_1 \left( \frac{1}{2} \right) = W$$

Substituting the value of $R_1$, we get

$$R + \frac{W}{3} \left( \frac{1}{2} \right) = W$$

Thus $R = W - \frac{W}{6} = \frac{5W}{6}$

(ii) Resolving horizontally, we get

$$\mu R = R_1 \sin 60^\circ \quad \Rightarrow \quad \mu R = R_1 \left( \frac{\sqrt{3}}{2} \right)$$

Substituting the values of $R$ and $R_1$, we get

$$\mu \left( \frac{5W}{6} \right) = \frac{W}{3} \left( \frac{\sqrt{3}}{2} \right)$$

i.e. $\mu = \frac{\sqrt{3}}{5}$
(i) The area of the shaded circle = \( \pi \left( \frac{7}{\pi} \right)^2 = 7 \) sq.units

The area of the rectangle OABF = \( 12 \times 6 = 72 \) sq.units

The area of the rectangle CDEF = \( 4 \times 6 = 24 \) sq.units

Hence the area of cross section of the concrete block = \( 72 + 24 - 7 = 89 \) sq. units

Let \( M \) be the mass per unit area and

let \( (\bar{x}, \bar{y}) \) be the centre of mass of the block with respect to the origin O.

The centre of mass of the circle and rectangles OABF and CDEF are at (3, 6); (6, 3) and (3, 8) respectively.

Applying the principle of moments in the \( x \) direction, we get

\[ 89M \bar{y} = 72Mg(6) + 24Mg(3) - 7Mg(3) = 483Mg \]

\[ \Rightarrow \bar{x} = \frac{483}{89} = 5.43 \]
Applying the principle of moments in the y direction, we get

\[ 89Mg\bar{y} = 72Mg(3) + 24Mg(8) - 7Mg(6) = 366Mg \]

\[ \Rightarrow \frac{\cancel{366}}{89} = 4.11 \]

(ii)

At equilibrium, the line through the centre of mass pass through O.

With respect to triangle OGH, where angle OHG is a right angle

\[ \tan \theta = \frac{\bar{x}}{\bar{y}} = \frac{5.43}{4.11} \quad \Rightarrow \quad \theta = \tan^{-1}\left(\frac{5.43}{4.11}\right) = 52.9^\circ \]
Question 7

(i)

Using the principle of conservation of momentum

$$500(5) - 200(7) = 500U_A + 200U_B$$

i.e. 

$$1100 = 500U_A + 200U_B$$

$$11 = 5U_A + 2U_B \ldots (i)$$

Using the law of Restitution,

$$\frac{3}{4} = \frac{U_B - U_A}{5 - (-7)} = \frac{U_B - U_A}{12}$$

i.e. 

$$U_B - U_A = 9 \ldots (ii)$$

(i) 

$$2U_B + 5U_A = 11 \Rightarrow$$

(ii) \times 2 

$$2U_B - 2U_A = 18$$

$$7U_A = -7 \Rightarrow U_A = -1 \text{ m/s}$$

Thus sphere A changes direction and travels to the left.

Substituting this value in (ii), we get $$U_B + 1 = 9$$ i.e. $$U_B = 8 \text{ m/s}$$

Consider the impact of sphere B with the wall.

Using the law of Restitution, 

$$\frac{1}{2} = \frac{U_B'}{U_B} \Rightarrow U_B' = \frac{U_B}{2} = \frac{8}{2} = 4 \text{ m/s}$$

Sphere B travels in the direction of A with speed of 4m/s, while sphere A travels with speed of 1m/s.

Thus there is surely going to be a second collision between the 2 spheres.
Using the law of Restitution,
\[
\frac{3}{4} = \frac{V_A - V_B}{4 - (1)} = \frac{V_A - V_B}{3} \quad \Rightarrow \quad V_A - V_B = \frac{9}{4} \quad \ldots (ii)
\]

Substituting in (i):
\[
5V_A + 2V_B = 13 \quad \Rightarrow \quad V_A = \frac{5}{2} \text{ m/s}
\]

\[
\text{Initial K.E.} = \frac{1}{2} \left( \frac{500}{1000} \right) (5)^2 + \frac{1}{2} \left( \frac{200}{1000} \right) (7)^2 = 11.15 \text{ J}
\]

\[
\text{Final K.E.} = \frac{1}{2} \left( \frac{500}{1000} \right) \left( \frac{5}{2} \right)^2 + \frac{1}{2} \left( \frac{200}{1000} \right) \left( \frac{1}{4} \right)^2 = 1.56875 \text{ J}
\]

\[
\text{Loss in K.E.} = 11.15 - 1.56878 = 9.58 \text{ J}
\]
Question 8

(i) Since \( \text{Power} = \text{Force} \times \text{Velocity} \)

Then \( 10,000 = F \times v \), where \( v \) is the velocity of the car.

\[ \Rightarrow \quad F = \frac{10000}{v} \]

When the car is going to descend, then \( 10,000 = F \times 2v \)

\[ \Rightarrow \quad F_i = \frac{10000}{2v} \]

Resolving forces along the line of greatest slope

At equilibrium and car ascending

\[ \frac{10000}{v} = R + 8000 \sin \theta \]

\[ \frac{10000}{v} = R + 8000 \left( \frac{1}{40} \right) = R + 200 \quad \text{...(i)} \]
At equilibrium and car descending

\[ \frac{10000}{2v} + 8000 \sin \theta = R \]

\[ \frac{10000}{2v} + 8000 \left( \frac{1}{40} \right) = R \quad \Rightarrow \quad R = \frac{10000}{2v} + 200 \quad \cdots (ii) \]

Substituting (ii) in (i), we get

\[ \frac{10000}{v} = \frac{10000}{2v} + 200 + 200 \]

This reduces to

\[ \frac{10000}{2v} = 400 \quad \Rightarrow \quad v = \frac{10000}{800} = 12.5 \text{ m/s} \]

∴ speed of ascent = 12.5 m/s

Substituting \( v \) in (i), we get

\[ \frac{10000}{12.5} = R + 200 \quad \Rightarrow \quad R = 600 \text{ N} \]

Applying Newton’s second law i.e. \( F = ma \), we get

\[ \frac{10000}{v} - R = 8000a \quad \Rightarrow \quad \frac{10000}{12.5} - 600 = 800a \]

Hence

\[ a = 0.25 \text{ m/s}^2 \]
Question 9

(a)

At equilibrium
Resolving perpendicular to the line of greatest slope
\[ R = mg \cos 30^\circ \] \text{...(i)}

Resolving along to the line of greatest slope
\[ \mu R = mg \sin 30^\circ \] \text{...(ii)}

\[
\frac{\mu R}{R} = \frac{mg \sin 30^\circ}{mg \cos 30^\circ} \Rightarrow \mu = \tan 30^\circ = \frac{1}{\sqrt{3}}
\]

(b)
At equilibrium

Resolving vertically: \( R \cos 30^\circ - \mu R \sin 30^\circ = mg \)

\[ \Rightarrow \quad R \left( \cos 30^\circ - \mu \sin 30^\circ \right) = mg \quad \ldots \text{(iii)} \]

Resolving horizontally.

Using \( F = ma = \frac{mv^2}{r} \) for motion in a circle

\[ \leftarrow \quad R \sin 30^\circ + \mu R \cos 30^\circ = \frac{mv^2}{r} \]

\[ \Rightarrow \quad R \left( \sin 30^\circ + \mu \cos 30^\circ \right) = \frac{mv^2}{r} \quad \ldots \text{(iv)} \]

\( \text{(iv)} \div \text{(iii)} \)

\[ \frac{\left( \sin 30^\circ + \mu \cos 30^\circ \right)}{\left( \cos 30^\circ - \mu \sin 30^\circ \right)} = \frac{v^2}{mg} \]

It simplifies to

\[ \frac{\left( 0.5 + \frac{1}{\sqrt{3}} \times 0.866 \right)}{\left( 0.866 - \frac{1}{\sqrt{3}} \times 0.5 \right)} = \frac{v^2}{600} \]

\[ \Rightarrow \quad v^2 = 1039.26 \]

i.e. \( v = 32.24 \text{m/s} \)
Question 10

(i)

Since the 7 rods are identical, then all the triangles are identical
i.e. they are all equilateral triangles.

By symmetry $C_{AD} = C_{CB}$; $T_{DE} = T_{CE}$ and $T_{AE} = T_{EB}$

As the external forces are at equilibrium

By symmetry $R_A = R_B$

And resolving vertically, we have $R_A + R_B = W + W + 4W$

$\Rightarrow 2R_B = 6W$ or $R_B = R_A = 3W$

(ii)

Consider the internal forces (Each joint is in equilibrium)

At A $\downarrow R_A = C_{AD} \sin 60^0 \Rightarrow 3W = C_{AD} \left( \frac{\sqrt{3}}{2} \right)$ or $C_{AD} = \frac{6W}{\sqrt{3}}$

$\therefore C_{CB} = C_{AD} = \frac{6W}{\sqrt{3}}$
At A \iff T_{AE} = C_{AD} \cos 60^0 \implies T_{AE} = \frac{6W}{\sqrt{3}} \left(\frac{1}{2}\right)

\therefore T_{EB} = T_{AE} = \frac{3W}{\sqrt{3}}

At E \updownarrow 2T_{DE} \sin 60^0 = 4W \implies 2T_{DE} \left(\frac{\sqrt{3}}{2}\right) = 4W

\therefore T_{EC} = T_{DE} = \frac{4W}{\sqrt{3}}

At D \iff C_{DC} = C_{AD} \cos 60^0 + T_{DE} \cos 60^0

C_{DC} = \left(\frac{6W}{\sqrt{3}} \left(\frac{1}{2}\right)\right) + \left(\frac{4W}{\sqrt{3}} \left(\frac{1}{2}\right)\right) = \frac{5W}{\sqrt{3}}

Hence rods AE and EB – we have a tension of magnitude \frac{3W}{\sqrt{3}}

rods AD and BC – we have a compression of magnitude \frac{6W}{\sqrt{3}}

rods DE and EC – we have a tension of magnitude \frac{4W}{\sqrt{3}}

rod DC – we have a compression of magnitude \frac{5W}{\sqrt{3}}